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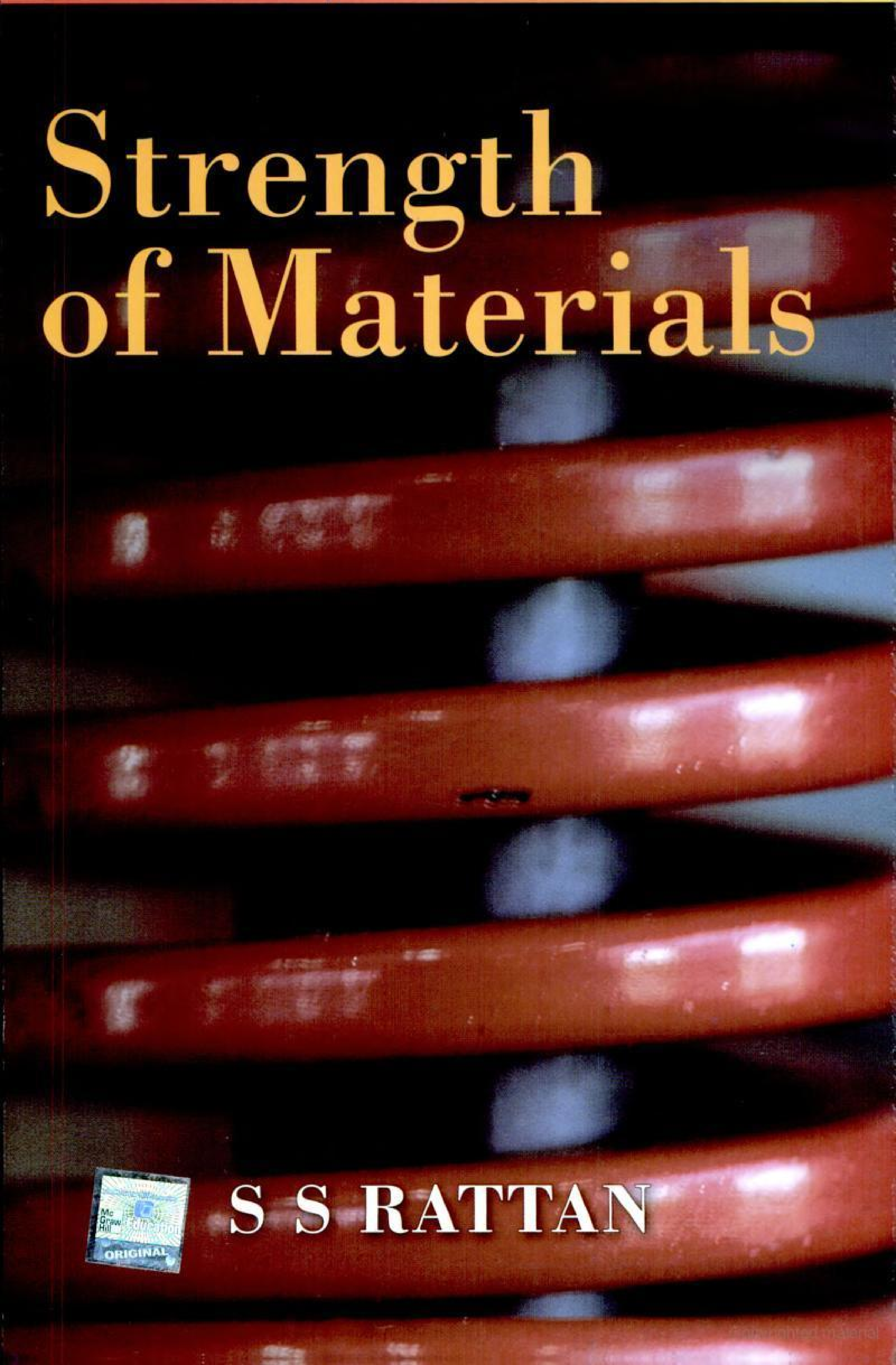

Strength of Materials

S S RATTAN

Tata McGraw-Hill

Published by the Tata McGraw-Hill Publishing Company Limited,
7 West Patel Nagar, New Delhi 110 008.

First reprint
RZLQCDDXRCYLQ

ISBN 13: 978-0-07-066895-9
ISBN 10: 0-07-066895-7

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Typeset at Bukprint India, B-180A, Guru Nanak Pura, Laxmi Nagar 110 092 and printed at Rashtriya Printers, M-135, Panchsheel Garden, Naveen Shahdara, Delhi-11 032

Cover: Rashtriya

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CONTENTS

PREFACE

An engineer always endeavours to design structural or machine members that are safe, durable and economical. To accomplish this, he has to evaluate the load-carrying capacity of the members so that they are able to withstand the various forces acting on them. The subject *Strength of Materials* deals with the strength, stability and rigidity of various structural or machine members such as beams, columns, shafts, springs, cylinders, etc. These days, a number of books on the subject are available in the market. However, it is observed that most of the books are feature-wise fine when considered on parameters like coverage of a topic, lucidity of writing, variety of solved and unsolved problems, quality of diagrams, etc., but usually, the students have to supplement a book with another book for one reason or the other. The present book aims to cover all good features in a single book.

The book is mainly aimed to be useful to degree-level students of mechanical and civil engineering as well as those preparing for AMIE and various other competitive examinations. However, diploma-level students will also find the book to be of immense use. The book will also benefit post-graduate students to some extent as it also contains some advance topics like bending of curved bars, rotating discs and cylinders, plastic bending and circular plates, etc. The salient features of the book are the following:

- A moderately concise and compact book covering all major topics
- Simple language to make it useful even to the average and weak students
- Logical and evolutionary approach in descriptions for better imagination and visualisation
- Physical concepts from simple and readily comprehensible principles
- Large number of solved examples
- Theoretical questions as well as sufficient number of unsolved problems at the end of each chapter
- Summary at the end of each chapter
- An appendix containing objective-type questions
- Another appendix containing important relations and results

It is expected that students using this book might have completed a course in applied mechanics. Chapters 1 and 2 introduce the concept of simple and compound stresses at a point. It is shown that an axial load may produce shear stresses along with normal stresses depending upon the section considered. The utility of Mohr's circle in transformation of stress at a point is also discussed. Chapter 3 explains the concept of strain energy that forms the basis of analysis in many cases. Chapters 4 to 8 are related to beams which may be simply supported, fixed at one or both ends or continuous having more than two supports. The analysis includes the computations of bending moment, shear force and bending and shear stresses under transverse loads. The concept of plastic deformations of beams beyond the elastic limit, being an advanced topic is taken up later and is discussed in Chapter 16. Sometimes, curved members such as rings and hooks are also loaded. Chapter 9 discusses the stresses developed in such members. The theory of torsion is developed in Chapter 10 which also includes its application to shafts transmitting power. The springs based on the same theory are discussed in the subsequent chapter. Columns are important members of structures. Chapter 12 discusses the equilibrium of columns and struts. However, the computation of stress in plane frame structures which is mostly included in the civil engineering curriculum is discussed later in Chapter 17. Some other important machine members include cylinders and spheres under internal or external pressures; flywheels, discs and cylinders which rotate while performing the required function; circular plates under concentrated or uniform loads. These topics are covered in chapters 13 to15. Chapter 18 discusses the properties of materials as well as the methods to determine the same.

Though students are expected to exert and solve the numerical problems given at the end of each chapter, hints to most of these are available at the publisher's website of the book for the benefit of average and weak students. However, full solutions of the unsolved problems are available to the faculty members at the same site. The facility can be availed by logging on to http://www.mhhe.com/rattan

In preparing the script, I relied heavily on the works of renowned authors whose writings are considered classics in the field. I am indeed indebted to them. I sincerely acknowledge the help of my many colleagues, who helped me in one form or the other in preparing this treatise. I also acknowledge the efforts of the editorial and production staff at Tata McGraw-Hill for taking pains in bringing out this book in an excellent format.

I am immensely thankful to the following reviewers who went through the manuscript and enriched it with their feedback.

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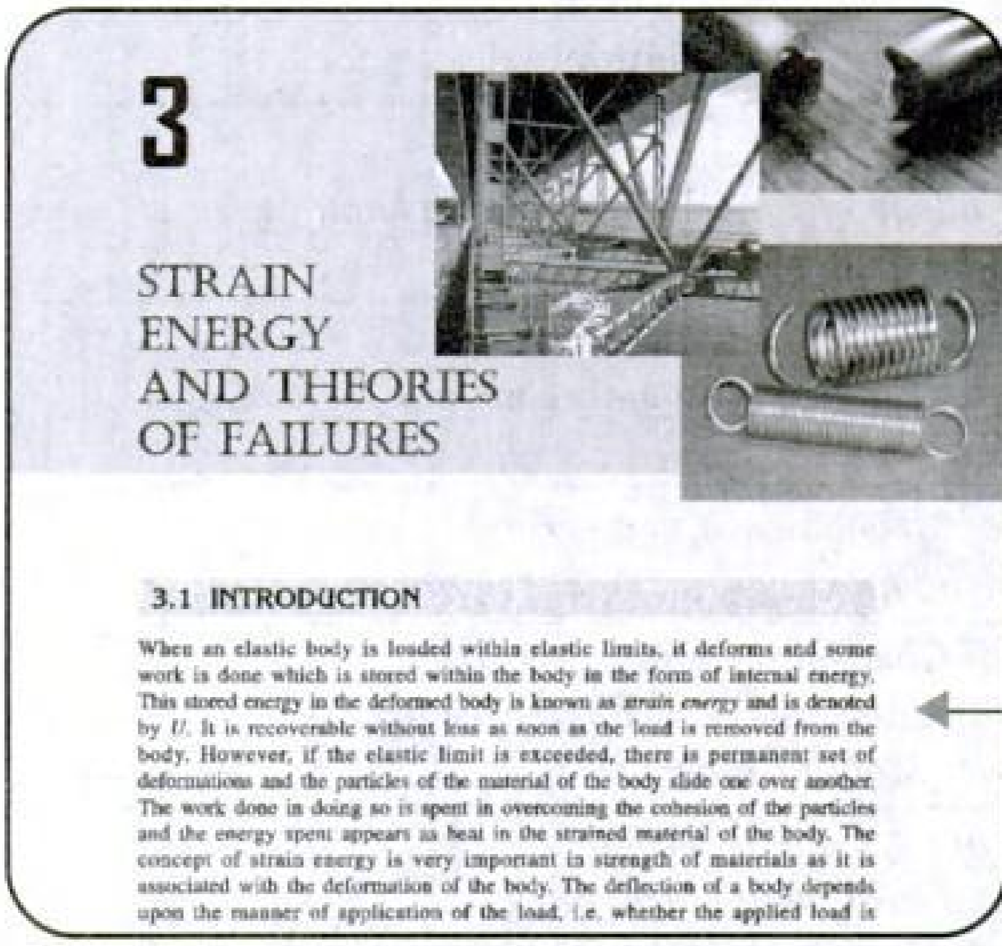
Finally, I am also indebted to my wife, Neena, and my children Ravneet and Jasmeet, for being patient with me while I went about the arduous task of preparing the manuscript. But for their sacrifice, I would not have been able to complete it in the most satisfying way.

A creation by a human being can never be perfect. A number of mistakes might have crept in the text. I shall be highly grateful to the readers and the users of the book for their uninhibited comments and pointing out the errors. Do feel free to contact me at ss_rattan@hotmail.com

S S Rattan

VISUAL WALKTHROUGH



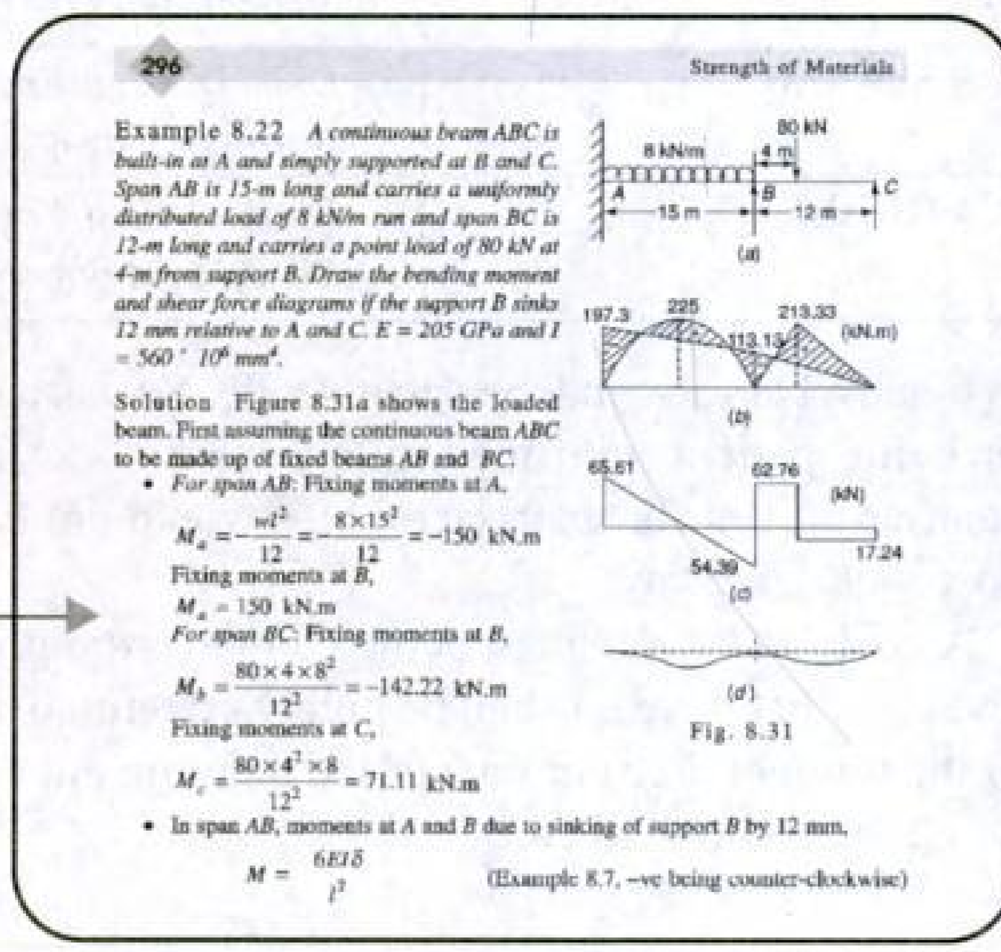
3

STRAIN ENERGY AND THEORIES OF FAILURES

3.1 INTRODUCTION

When an elastic body is loaded within elastic limits, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as *strain energy* and is denoted by *U*. It is recoverable without loss as soon as the load is removed from the body. However, if the elastic limit is exceeded, there is permanent set of deformations and the particles of the material of the body slide one over another. The work done in doing so is spent in overcoming the cohesion of the particles and the energy spent appears as heat in the strained material of the body. The concept of strain energy is very important in strength of materials as it is associated with the deformation of the body. The deflection of a body depends upon the manner of application of the load, i.e. whether the applied load is

Introduction at the beginning of each chapter sums up the aim and contents of the chapter.



296 Strength of Materials

Example 8.22 *A continuous beam ABC is built-in at A and simply supported at B and C. Span AB is 15-m long and carries a uniformly distributed load of 8 kN/m run and span BC is 12-m long and carries a point load of 80 kN at 4-m from support B. Draw the bending moment and shear force diagrams if the support B sinks 12 mm relative to A and C. E = 205 GPa and I = 560 × 10⁶ mm⁴.*

Solution Figure 8.31*a* shows the loaded beam. First assuming the continuous beam *ABC* to be made up of fixed beams *AB* and *BC*.

- *For span AB*: Fixing moments at *A*,

$$M_a = -\frac{wl^2}{12} = -\frac{8\times 15^2}{12} = -150 \text{ kN.m}$$

Fixing moments at *B*,

$M_a = 150$ kN.m

For span BC: Fixing moments at *B*,

$$M_b = \frac{80\times 4\times 8^2}{12^2} = -142.22 \text{ kN.m}$$

Fixing moments at *C*,

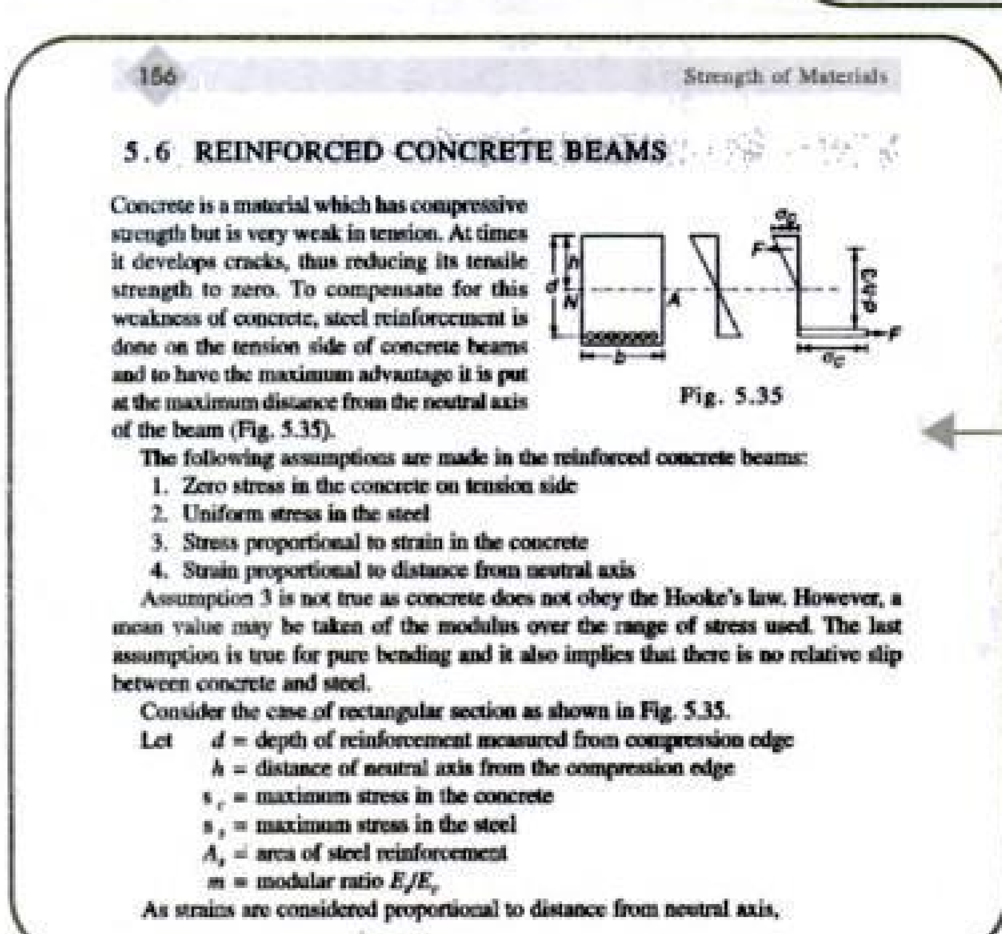
$$M_c = \frac{80\times 4^2\times 8}{12^2} = 71.11 \text{ kN.m}$$

- In span *AB*, moments at *A* and *B* due to sinking of support *B* by 12 mm,

$$M = \frac{6EI\delta}{l^2}$$ (Example 8.7, –ve being counter-clockwise)

Fig. 8.31

A variety of solved examples to reinforce the concepts.



156 Strength of Materials

5.6 REINFORCED CONCRETE BEAMS

Concrete is a material which has compressive strength but is very weak in tension. At times it develops cracks, thus reducing its tensile strength to zero. To compensate for this weakness of concrete, steel reinforcement is done on the tension side of concrete beams and to have the maximum advantage it is put at the maximum distance from the neutral axis of the beam (Fig. 5.35).

Fig. 5.35

The following assumptions are made in the reinforced concrete beams:

1. Zero stress in the concrete on tension side
2. Uniform stress in the steel
3. Stress proportional to strain in the concrete
4. Strain proportional to distance from neutral axis

Assumption 3 is not true as concrete does not obey the Hooke's law. However, a mean value may be taken of the modulus over the range of stress used. The last assumption is true for pure bending and it also implies that there is no relative slip between concrete and steel.

Consider the case of rectangular section as shown in Fig. 5.35.

Let d = depth of reinforcement measured from compression edge
h = distance of neutral axis from the compression edge
s_c = maximum stress in the concrete
s_s = maximum stress in the steel
A_s = area of steel reinforcement
m = modular ratio E_s/E_c

As strains are considered proportional to distance from neutral axis,

Concise and comprehensive treatment of topics with emphasis on fundamental concepts.

Bending Stress in Beams 181

Review Questions

1. What do you mean by the terms neutral axis and neutral surface?
2. Develop the theory of simple bending, clearly stating the assumptions made.
3. Prove the relation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$ for simple bending.
4. Define the term *moment of resistance*.
5. What do you mean by the term flitched beams? Develop a relation for the moment of resistance for such a beam.
6. What are reinforced concrete beams? Where are they used? How do you find the moment of resistance of such beams?
7. What are meant by the terms *principal axis, principal moments of inertia* and *product of inertia*?
8. How do you find the *principal moments of inertia* of an area?
9. Describe the method to find the moments of inertia about two mutually perpendicular axes through the centroid when the moments of inertia about the principal axes are known.
10. What is the middle third rule for rectangular sections?
11. A 40-kN load acts on a short column of 80 mm × 60 mm rectangular cross-section at a point 30 mm from the shorter side and 15 mm from the longer side. Find the maximum tensile and compressive stresses in the section. (10.4 MPa; 27.1 MPa)
12. A hollow circular bar used as a beam has an outside diameter twice of the inside diameter. If it is subjected to a maximum bending moment of 40 kN.m and the allowable bending stress is 100 MPa, determine the inside diameter of the bar. (81.6 mm)

A number of theoretical questions and unsolved problems for practice to widen the horizon of comprehension of the topic.

Summary at the end of each chapter recapitulates the inferences for quick revision.

Slope and Deflection 259

7.11 BETTI'S THEOREM OF RECIPROCAL DEFLECTIONS

It may be stated as follows:

In an elastic system, the external work done by a force acting at P during the deflections caused by another force at Q is equal to the external work done by the force at Q during the deflections caused by the force at P.

In the mathematical form,

$$F_p \delta_{pq} = Q_p \delta_{qp}$$

Summary

- Excessive deflections can cause visible or invisible cracks in beams. Also, excessive deflections perceptible by naked eye give a feeling of unsafe structure to the occupants of a building causing adverse effect on their health.
- The designing of a beam from deflection aspect is known as *stiffness criterion.*
- Deflection profile of a beam is known as its *elastic curve*.
- Governing *differential equation* of a beam under the action of bending moment is $EI(d^2y/dx^2) = M$
- Main methods to find the slope and deflection of a beam are double integration method, Macaulay's method, area-moment method, strain energy method and conjugate beam method.
- In double integration method, the equation of the elastic curve is integrated twice to obtain the deflection of the beam at any cross-section. The constants of integration are found by applying the end conditions.
- In Macaulay's method a single equation is written for the bending moment for

90 Strength of Materials

Example 3.7 *A lift is operated by three ropes each having 28 wires of 1.4 mm diameter. The cage weighs 1.2 kN and the weight of the rope is 4.2 N/m length. Determine the maximum load carried by the lift if each wire is of 36 m length and the lift operates (i) without any drop (ii) with a drop of 96 mm during operations. E (rope) = 72 GPa and allowable stress = 115 MPa*

Solution Total area of cross section, $A = \frac{\pi}{4}.(1.4)^2 \times 3 \times 28 = 129.3$ mm²

The maximum stress occurs at the top of the wire rope where the weight of the rope is maximum.

Thus maximum load = weight of cage + weight of rope
= 1200 + 3 × 36 × 4.2 = 1653.6 N

Initial stress in the rope, $\sigma = \frac{1653.6}{129.3} = 12.8$ MPa

Equivalent static stress available for carrying the load = 115 − 12.8 = 102.2 MPa

Thus, equivalent static load that can be carried,

$P_e = 102.2 \times 129.3 = 13\,214$ N

The extension of the rope, $\Delta = \frac{102.2 \times 36\,000}{72\,000} = 51.1$ mm

(i) With no drop, Let *W* be the weight which can be applied suddenly, $W.\Delta = \frac{1}{2}P.\Delta$

or $W = 13214/2 = 6607$ N or 6.607 kN

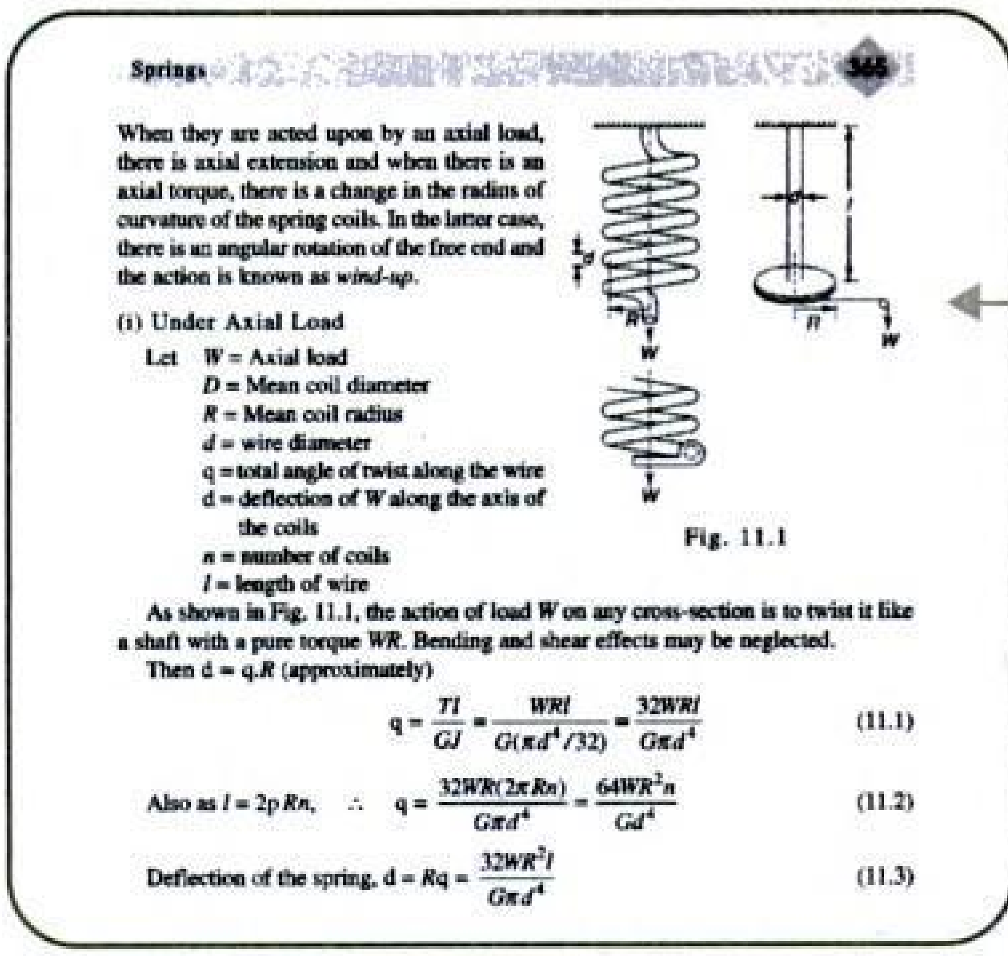
(ii) With 96 mm drop, Let *W* be the weight,

$$W(h + \Delta) = \frac{1}{2}P.\Delta \quad \text{or} \quad W(96 + 51.1) = \frac{1}{2} \times 13214 \times 51.1$$

or $W = 2295$ N or 2.295 kN

Example 3.8 *A vertical composite tie bar rigidly fixed at the upper end consists*

International system of units (SI) throughout the book for universal approval.



Springs 365

When they are acted upon by an axial load, there is axial extension and when there is an axial torque, there is a change in the radius of curvature of the spring coils. In the latter case, there is an angular rotation of the free end and the action is known as *wind-up*.

(i) Under Axial Load

Let W = Axial load
D = Mean coil diameter
R = Mean coil radius
d = wire diameter
q = total angle of twist along the wire
d = deflection of W along the axis of the coils
n = number of coils
l = length of wire

Fig. 11.1

As shown in Fig. 11.1, the action of load W on any cross-section is to twist it like a shaft with a pure torque WR. Bending and shear effects may be neglected.

Then d = q.R (approximately)

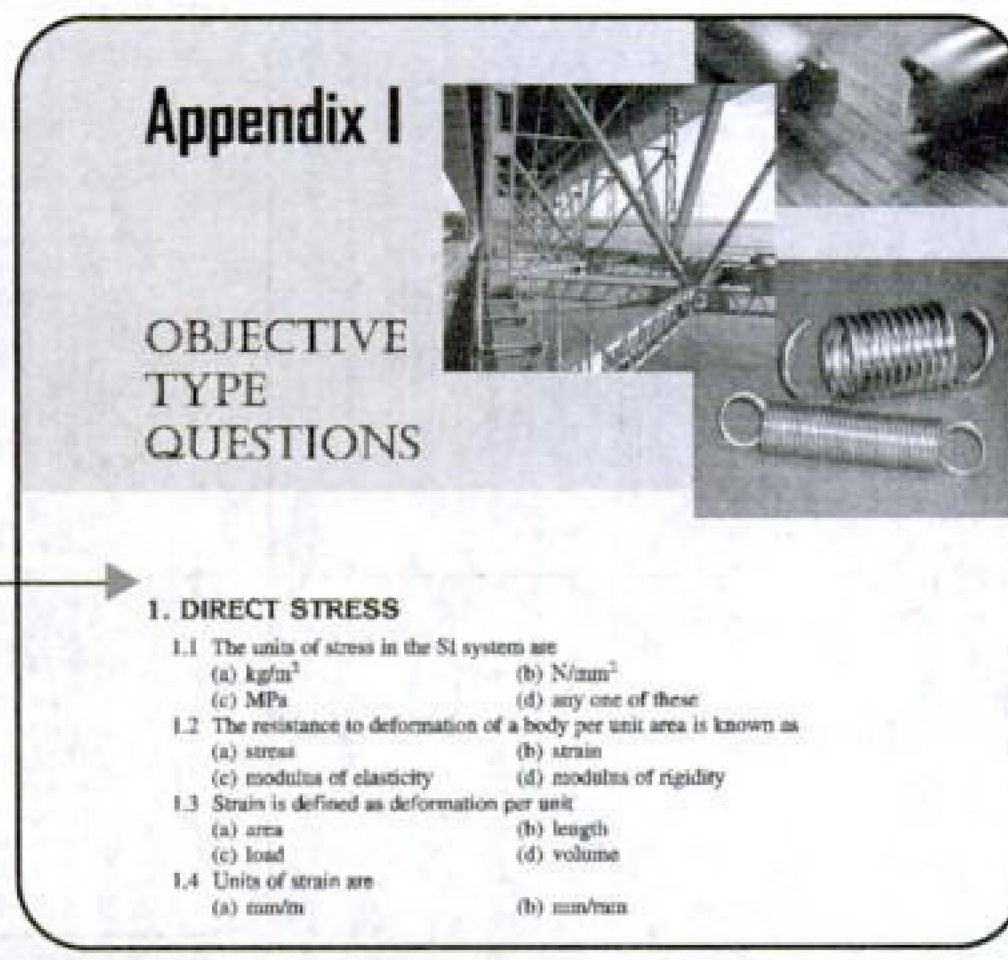
$$q = \frac{Tl}{GJ} = \frac{WRl}{G(\pi d^4/32)} = \frac{32WRl}{G\pi d^4} \quad (11.1)$$

Also as $l = 2pRn$, $\therefore$ $$q = \frac{32WR(2\pi Rn)}{G\pi d^4} = \frac{64WR^2n}{Gd^4} \quad (11.2)$$

Deflection of the spring, $$d = Rq = \frac{32WR^2l}{G\pi d^4} \quad (11.3)$$

Simple diagrams for easy visualization of the explanations.

Appendix containing multiple choice questions to prepare for competitive examinations.



Appendix I

OBJECTIVE TYPE QUESTIONS

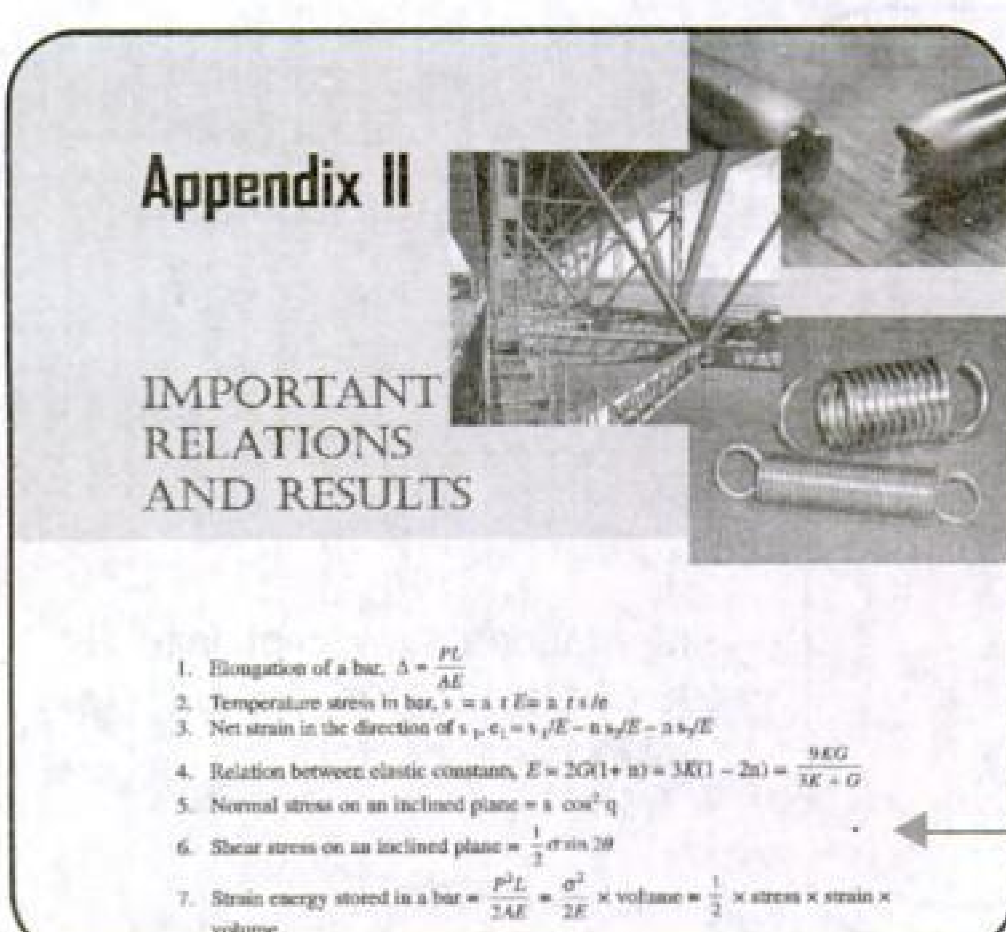
1. DIRECT STRESS

1.1 The units of stress in the SI system are
(a) kg/m^2 (b) N/mm^2
(c) MPa (d) any one of these

1.2 The resistance to deformation of a body per unit area is known as
(a) stress (b) strain
(c) modulus of elasticity (d) modulus of rigidity

1.3 Strain is defined as deformation per unit
(a) area (b) length
(c) load (d) volume

1.4 Units of strain are
(a) mm/m (b) mm/mm



Appendix II

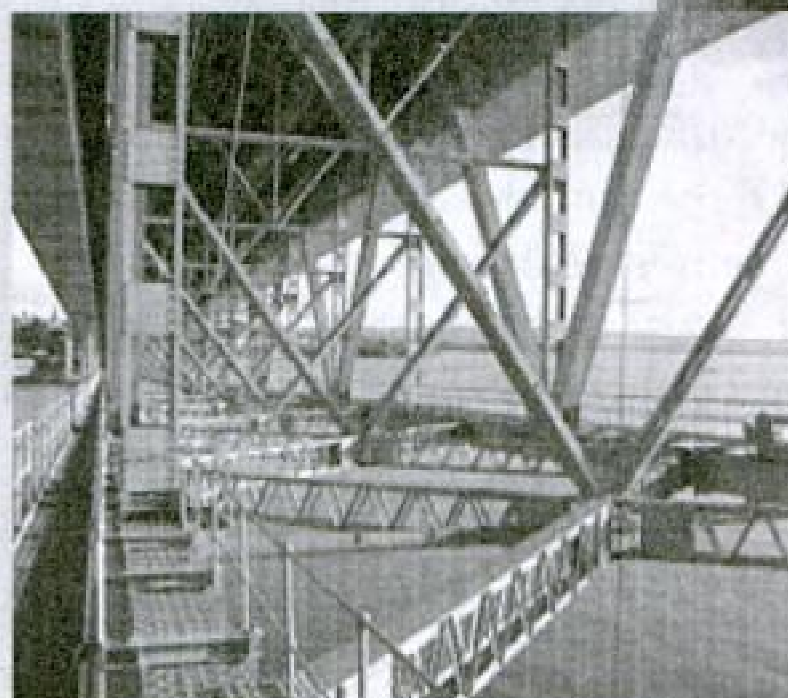
IMPORTANT RELATIONS AND RESULTS

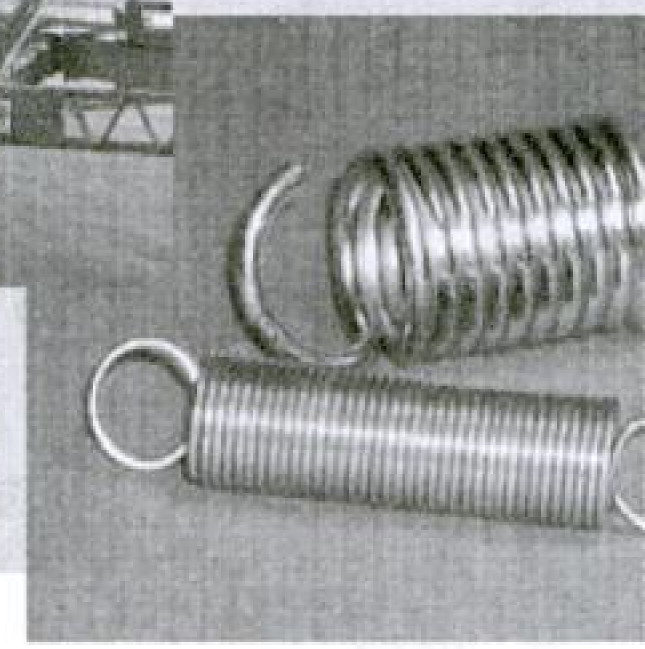
1. Elongation of a bar, $\Delta = \frac{PL}{AE}$
2. Temperature stress in bar, s = a t E = a t s /e
3. Net strain in the direction of s_1, $e_1 = s_1/E - a\,s_2/E - a\,s_3/E$
4. Relation between elastic constants, $E = 2G(1+ n) = 3K(1 - 2n) = \frac{9KG}{3K + G}$
5. Normal stress on an inclined plane = s $\cos^2$ q
6. Shear stress on an inclined plane = $\frac{1}{2}\sigma \sin 2\theta$
7. Strain energy stored in a bar = $\frac{P^2L}{2AE} = \frac{\sigma^2}{2E}$ × volume = $\frac{1}{2}$ × stress × strain × volume

Appendix containing important relations for ready reference.

LIST OF SYMBOLS

a	area
A	area
b	width
B	width
d	diameter, height, depth
D	diameter
e	eccentricity
E	modulus of elasticity, Young's modulus
F	force
g	acceleration due to gravity
G	shear modulus, modulus of rigidity
h	height, distance
H	height
I	moment of inertia
l	length
j	number of joints
J	polar moment of inertia
k	torsional stiffness, stiffness of spring
K	Bulk modulus
l	length
L	length, load factor
m	mass, modular ratio, number of members
M	moment, bending moment, mass
n	number of coils
p	pressure, compressive stress
P	force, load
q	shear flow
r	radius
R	radius, reaction
s	length
S	shape factor
t	thickness, time, temperature
T	torque
u	strain energy density
U	strain energy, resilience
V	volume
w	rate of loading
W	force, weight, load
x, y, z	rectangular coordinates, distances
Z	section modulus
σ	direct stress
θ	angle
φ	angle, shear strain
α	angle, coefficient of thermal expansion
δ	increment of quantity, deflection, extension
τ	shear stress
Δ	elongation
ε	direct strain
π	3.1416
ν	Poisson's ratio
ψ	angle
γ	angle
ω	angular velocity
ρ	density





1

SIMPLE STRESS AND STRAIN

1.1 INTRODUCTION

External forces acting on individual structural or machine members of an engineering design are common. An engineer always endeavours to have such a design so that these are safe, durable and economical. Thus load carrying capacity of the members being designed is of paramount importance to know their dimensions to have the minimum cost. The subject *Strength of Materials* deals with the strength or the load-carrying capacity of various members such as beams and columns. It also considers their stability and rigidity. *Theory of structures* involves the application of these principles to structures made up of beams, columns, slabs and arches.

Force acting on a body is termed as *load*. A *concentrated load* is also known as a *point load* and a distributed load over a length is known as *distributed load*. Distributed load of constant value is called *uniformly distributed load*. If a structure as a whole is in equilibrium, its members are also in equilibrium individually which implies that the resultant of all the forces acting on a member must be zero. However, the forces acting on a body tend to deform or torn the body. For example, a load P acting on a body tends to pull it apart (Fig. 1.1*a*). This type of pull may also be applied if one end of the body is fixed (Fig. 1.1*b*). In this case, the balancing force is provided by the reaction of the fixed end. Such type of pulling force is known as *tension* or *tensile force*. A tensile force tends to increase the length and decrease the cross-section of the body.

In a similar way, a force tending to push or compress a body is known as *compression* or *compressive force* which tends to shorten the length (Fig. 1.1*c*).

Fig. 1.1

Usually, the forces acting on a body along the longitudinal axis are known as *direct* or *axial forces* and the forces acting normal to the longitudinal axis of a body are known as *transverse* or *normal* forces.

In the elementary theory of analysis, a material subjected to external forces is assumed to be perfectly elastic, i.e. the deformations caused to the body totally disappear as soon as the load or forces are removed. Other assumptions are that the materials are *isotropic* (same properties in all directions) and *homogeneous* (same properties anywhere in the body).

1.2 STRESS

The applied external forces on a body are transmitted to the supports through the material of the body. This phenomenon tends to deform the body and causes it to develop equal and opposite internal forces. These *internal forces* by virtue of cohesion between particles of the material tend to resist the deformation. The magnitude of the internal resisting forces is equal to the applied forces but the direction is opposite.

Let the member shown in Fig. 1.1*a* be cut through the section *X-X* as shown in Fig. 1.2. Now, each segment of the member is in equilibrium under the action of force *P* and the internal resisting force. The resisting force per unit area of the surface is known as *intensity of stress* or simply *stress* and is denoted by σ. Thus if the load *P* is assumed as uniformly distributed over a sectional area *A*, then the stress σ is given by

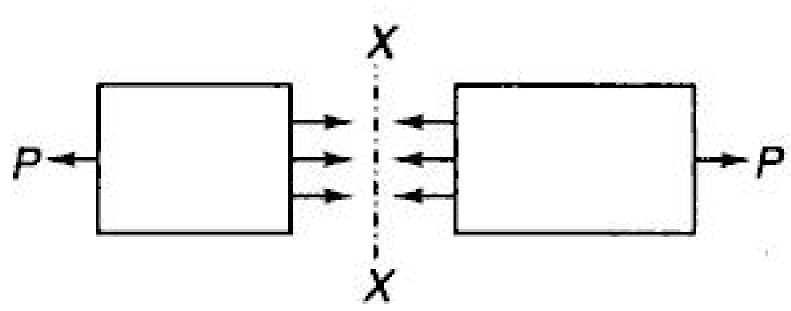


Fig. 1.2

$$\sigma = P/A \tag{1.1}$$

However, if the intensity of stress is not uniform throughout the body, then the stress at any point is defined as

$$\sigma = \delta P/\delta A$$

where δA = Infinitesimal area of cross-section

and δP = Load applied on area δA

The stress may be tensile or compressive depending upon the nature of forces applied on the body.

Stress at the elastic limit is usually referred as *proof stress*.

Units

The unit of stress is N/m^2 or Pascal (Pa). However, this is a very small unit, almost the stress due to placing an apple on an area of 1 m^2. Thus it is preferred to express stress in units of MN/m^2 or MPa.

$$1\ MN/m^2 = 1\ MPa = 1 \times 10^6\ N/m^2 = 1\ N/mm^2$$

Also $$1\ GPa = 1000\ MPa = 1000\ N/mm^2 = 1\ kN/mm^2$$

In numerical problems, it is always convenient to express the units of stress mentioned in MPa and GPa in the form of N/mm^2.

1.3 SHEAR STRESS

When two equal and opposite parallel forces not in the same line act on two parts of a body, then one part tends to slide over or shear from the other across any section and the stress developed is termed as *shear stress*. In Fig. 1.3*a* and *b*, the material is sheared along any section *X-X* whereas in a riveted joint (Fig.1.3*c*), the shearing is across the rivet diameter.

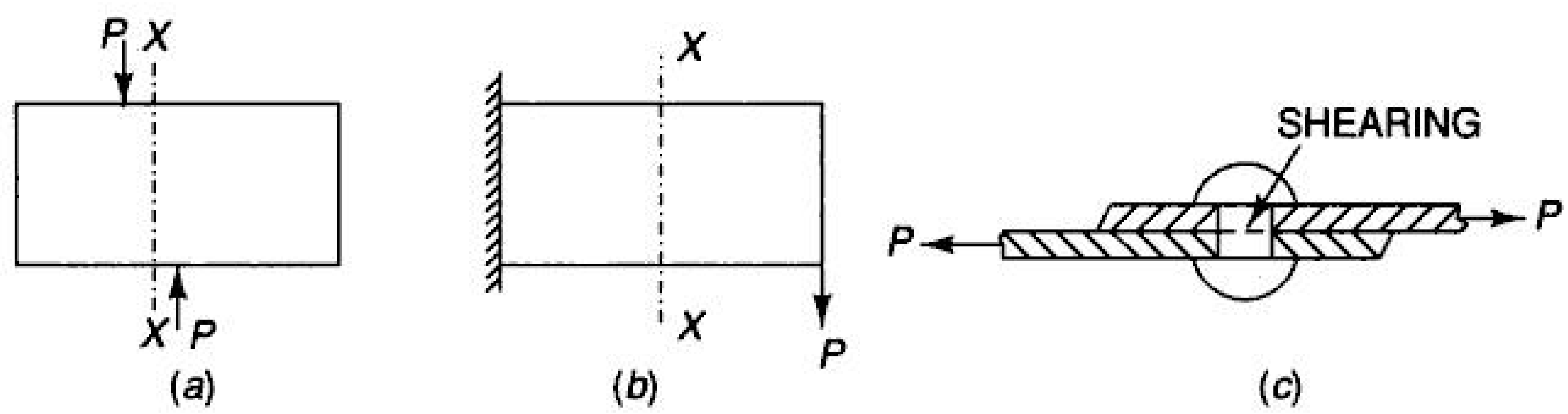


Fig. 1.3

If P is the force applied and A is the area being sheared, then the intensity of shear stress is given by

$$\tau = P/A \tag{1.2}$$

and if the intensity of shear stress varies over an area,

$$\tau = \delta P/\delta A$$

Remember that shear stress is always tangential to the area over which it acts.

Complimentary Shear Stress

Consider an infinitely small rectangular element *ABCD* under shear stress of intensity τ acting on planes *AD* and *BC* as shown in Fig. 1.4*a*. It is clear from the figure that the shear stress acting on the element will tend to rotate the block in the clockwise direction.

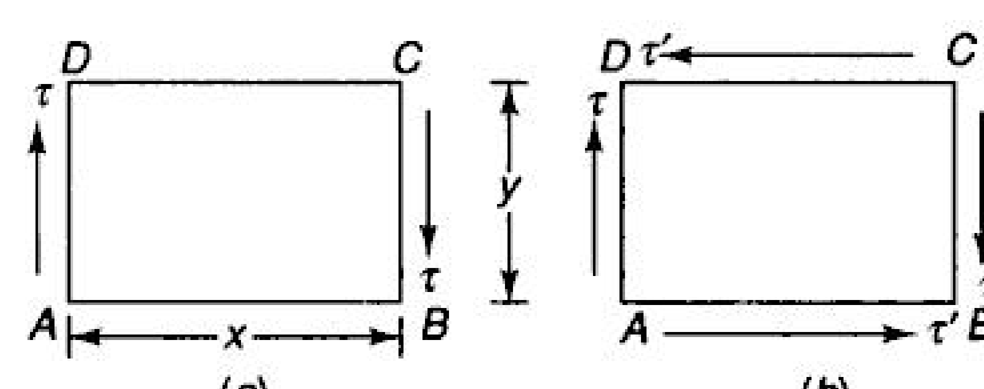


Fig. 1.4

As there is no other force acting on the element, static equilibrium of the element can only be attained if another couple of the same magnitude is applied in the counter-clockwise direction. This can be achieved by having shear stress of intensity τ' on the faces *AB* and *CD* (Fig. 1.4*b*).

Assuming x and y to be the lengths of the sides *AB* and *BC* of the rectangular element and a unit thickness perpendicular to the figure,

The force of the given couple = $\tau.(y.1)$

The moment of the given couple = $(\tau.y).x$

Similarly,

The force of balancing couple = $\tau'.(x.1)$

The moment of balancing couple = $(\tau'.x).y$

For equilibrium, equating the two,

$$(\tau.y).x = (\tau'.x).y \quad \text{or} \quad \tau = \tau'$$

which shows that the magnitude of the balancing shear stresses is the same as of the applied stresses. The shear stresses on the transverse pair of faces are known as *complimentary shear stresses*. Thus every shear stress is always accompanied by an equal complimentary shear stress on perpendicular planes.

Owing to the characteristic of complimentary shear stresses for the equilibrium of members subjected to shear stresses, near a free boundary on which no external force acts, the shear stress must follow a direction parallel to the boundary. This is because any component of the shear force perpendicular to the surface will find no complimentary shear stress on the boundary plane. The presence of complimentary shear stress may cause an early failure of *anisotropic materials* such as timber which is weaker in shear along the grain than normal to the grain.

1.4 STRAIN

The deformation of a body under a load is proportional to its length. To study the behaviour of a material, it is convenient to study the deformation per unit length of a body than its total deformation. The elongation per unit length of a body is known as *strain* and is denoted by Greek symbol ε. If Δ is the elongation of a body of length L, the strain ε is given by

$$\varepsilon = \Delta/L \tag{1.3}$$

As it is a ratio of similar quantities, it is dimensionless.

Shear Strain

A rectangular element of a body is distorted by shear stresses as shown in Fig. 1.5. If the lower surface is assumed to be fixed, the upper surface slides relative to the lower surface and the corner angles are altered by angle φ. *Shear strain* is defined as the change in the right angle of the element measured in radians and is dimensionless.

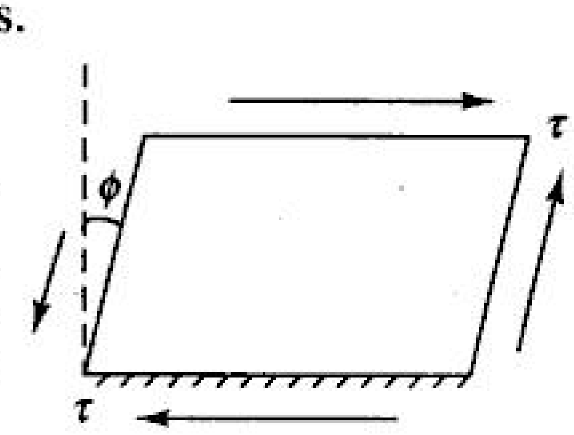


Fig. 1.5

1.5 MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

For elastic bodies, the ratio of stress to strain is constant and is known as *Young's Modulus* or the *Modulus of Elasticity* and is denoted by E, i.e.,

$$E = \sigma/\varepsilon \tag{1.4}$$

Strain has no units as it is a ratio. Thus E has the same units as stress.

The materials that maintain this ratio are said to obey *Hook's law* which states that within elastic limits, strain is proportional to the stress producing it. The elastic limit of a material is determined by plotting a tensile test diagram (Refer section 1.15).

Young's modulus is the stress required to cause a unit strain. As a unit strain means elongation of a body equal to original length (since $\varepsilon = \Delta/L$), this implies that Young's modulus is the stress or the force required per unit area to elongate the body by its original size or to causes the length to be doubled. However, for most of the engineering materials, the strain does not exceed 1/1000. Obviously, mild steel has a much higher value of Young's modulus E as compared to rubber.

Similarly, for elastic materials, the shear strain is found to be proportional to the applied shear stress within the elastic limit. *Modulus of rigidity* or *shear modulus* denoted by G is the ratio of shear stress to shear strain, i.e.

$$G = \tau/\varphi \tag{1.5}$$

1.6 ELONGATION OF A BAR

An expression for the elongation of a bar of length L and cross-sectional area A under the action of a force P is obtained below:

$$\text{As} \quad E = \frac{\sigma}{\varepsilon} \quad \therefore \quad \varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \frac{\Delta}{L} = \frac{P}{AE}$$

Thus elongation of a bar of length L, $\quad \Delta = \dfrac{PL}{AE}$ (1.6)

1.7 PRINCIPLE OF SUPERPOSITION

The principle of superposition states that if a body is acted upon by a number of loads on various segments of a body, then the net effect on the body is the sum of the effects caused by each of the loads acting independently on the respective segment of the body. Thus each segment can be considered for its equilibrium. This is done making a diagram of the segment alongwith various forces acting on it. This diagram is generally referred as *free body diagram*. The principle of superposition is applicable to all parameters like stress, strain and deflection. However, it is not applicable to materials with non-linear stress-strain characteristics which do not follow Hook's law.

Example 1.1 *A steel bar of 25-mm diameter is acted upon by forces as shown in Fig. 1.6a. What is the total elongation of the bar? Take E = 190 GPa.*

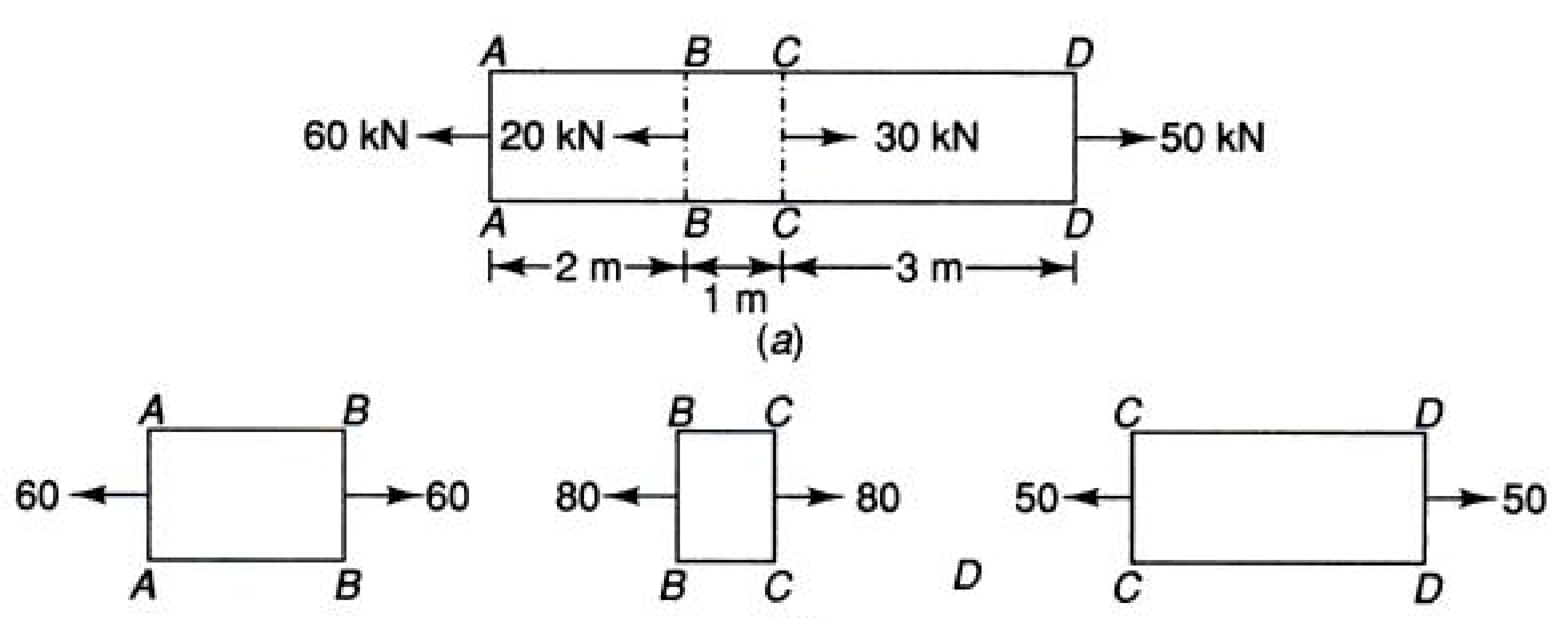


Fig. 1.6

Solution Area of the section = $\frac{\pi}{4}(25)^2 = 490.88\text{ mm}^2$, $E = 190$ GPa $= 190\,000\text{ N/mm}^2$

Forces in various segments are considered by taking free-body diagram of each segment as follows (Fig. 1.6*b*):

Segment AB: At section *AA*, it is 60 kN tensile and for force equilibrium of this segment, it is to be 60 kN tensile at *BB* also.

Segment BC:

Force at section *BB* = 60 kN (as above) + 20 kN (tensile force at section *BB*)
= 80 kN (tensile) = Force at section *CC*

Segment CD:

Force at section *CC* = 80 kN (as above) – 30 kN (compressive force at section *CC*)
= 50 kN (tensile) = Force at section *DD*

Elongation is given by, $\Delta = \frac{PL}{AE}$

$$= \frac{1}{490.88 \times 190\,000}(60\,000 \times 2000 + 80\,000 \times 1000 + 50\,000 \times 3000) = 3.75\text{ mm}$$

Example 1.2 *A steel circular bar has three segments as shown in Fig. 1.7a. Determine*

(i) the total elongation of the bar
(ii) the length of the middle segment to have zero elongation of the bar
(iii) the diameter of the last segment to have zero elongation of the bar
Take E = 205 GPa.

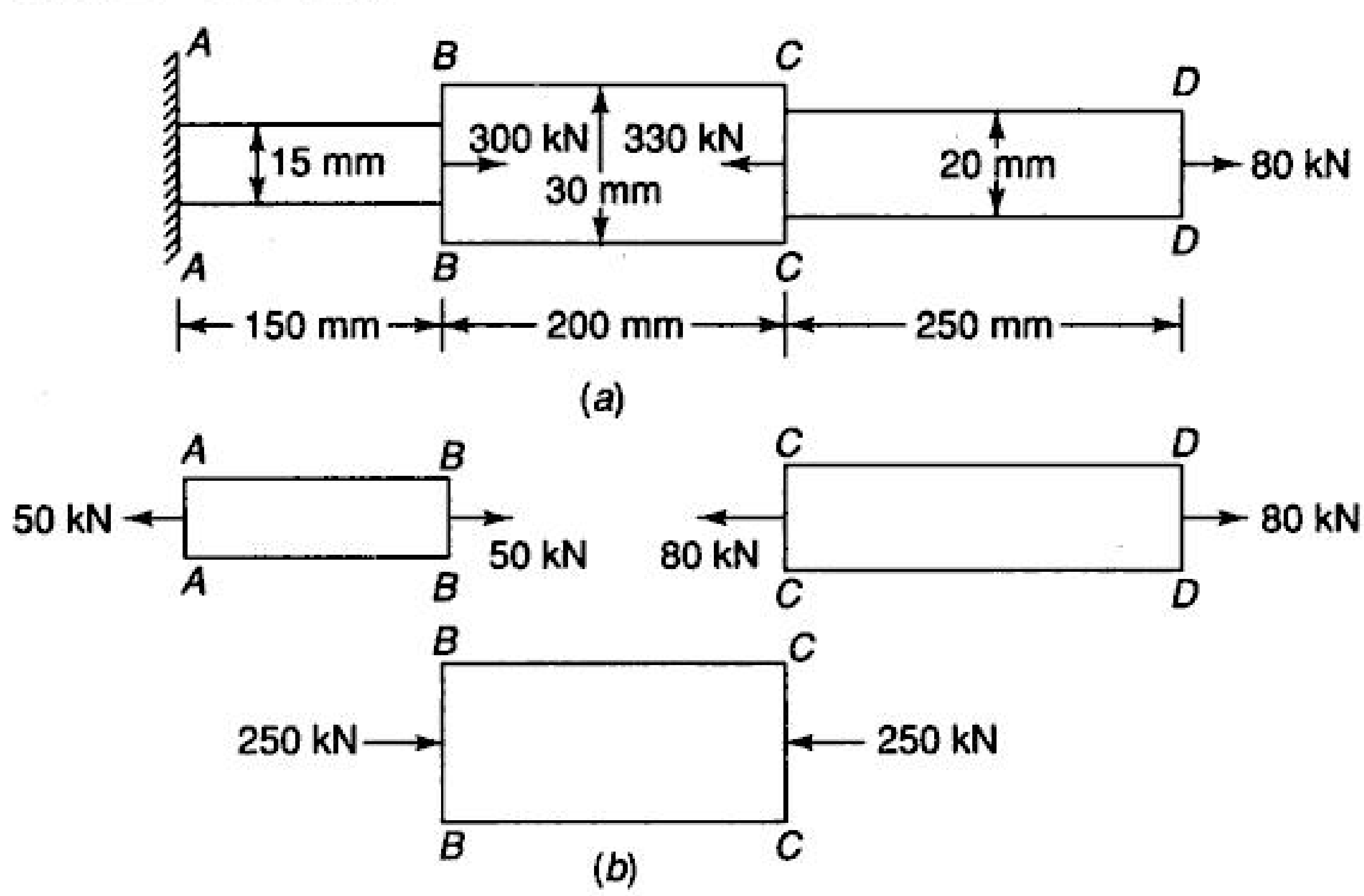


Fig. 1.7

Solution Forces in various segments (Fig. 1.7*b*):

(i) *Segment CD*: At section *DD*, it is 80 kN tensiles and for force equilibrium of this segment, at *CC* also it is to be 80 kN tensile.

Segment BC:

Force at section CC = 80 kN (as above) – 330 kN (compressive force at section CC)
= – 250 kN (compressive) = Force at section BB

Segment AB:

Force at section BB = – 250 kN (as above) + 300 kN (tensile force at section BB)
= 50 kN (tensile) = Force at section AA

Total elongation,

$$\Delta = \frac{1}{(\pi/4) \times 205\,000}\left(\frac{50\,000 \times 150}{15^2} - \frac{250\,000 \times 200}{30^2} + \frac{80\,000 \times 250}{20^2}\right)$$

$$= \frac{1}{161\,007}(33\,333.3 - 55\,555.5 + 50\,000) = 0.173 \text{ mm}$$

(ii) Let the length of the middle segment be L to have zero elongation of the bar.

Then $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - \frac{250\,000 \times L}{30^2} + 50\,000\right) = 0$$

or $$L = \frac{30^2}{250\,000} \times 83\,333.3 = 300 \text{ mm}$$

(iii) Let the diameter of the last segment be d to have zero elongation of the bar.

$\therefore$ $$\Delta = \frac{1}{161\,007}\left(33\,333.3 - 55\,555.5 + \frac{80\,000 \times 250}{d^2}\right) = 0$$

$$d^2 = \frac{80\,000 \times 250}{22\,222.2} = 900 \quad \text{or} \quad d = 30 \text{ mm}$$

1.8 BARS OF TAPERING SECTION

Bars of tapering section can be of conical section or of trapezoidal section with uniform thickness.

Conical Section

Consider a bar of conical section under the action of axial force P as shown in Fig. 1.8.

Let D = diameter at the larger end

d = diameter at the smaller end

L = length of the bar

E = Young's modulus of the bar material

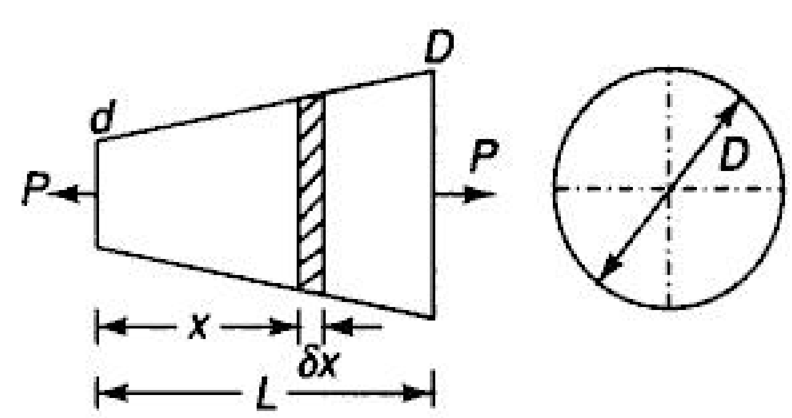


Fig. 1.8

Consider a very small length δx at a distance x from the small end.

The diameter at a distance x from the small end $= d + \dfrac{D - d}{L}.x$

The extension of a small length

$$= \frac{P.\delta x}{\frac{\pi}{4}\left(d + \frac{D-d}{L}x\right)^2 .E} \quad \ldots\ldots \left(\Delta = \frac{PL}{AE}\right)$$

$$\text{Extension of the whole rod} = \int_0^L \frac{4P}{\pi\left(d + (D-d)x/L\right)^2 .E}.dx$$

$$= \frac{4P}{\pi E}\int_0^L \left(d + \frac{D-d}{L}.x\right)^{-2}.dx = -\frac{4P}{\pi E}.\frac{L}{(D-d)}\left(\frac{1}{(d + (D-d)x/L)}\right)_0^L$$

$$= \frac{4PL}{\pi E(D-d)}\left(\frac{1}{d} - \frac{1}{D}\right) = \frac{4PL}{\pi E(D-d)}\left(\frac{D-d}{dD}\right) = \frac{4PL}{\pi EdD} \qquad (1.7)$$

Trapezoidal Section of Uniform Thickness

Let

B = width at the larger end $\qquad$ b = width at the smaller end

t = thickness of the section $\qquad$ L = length of the bar

E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end of the rod (Fig.1.9).

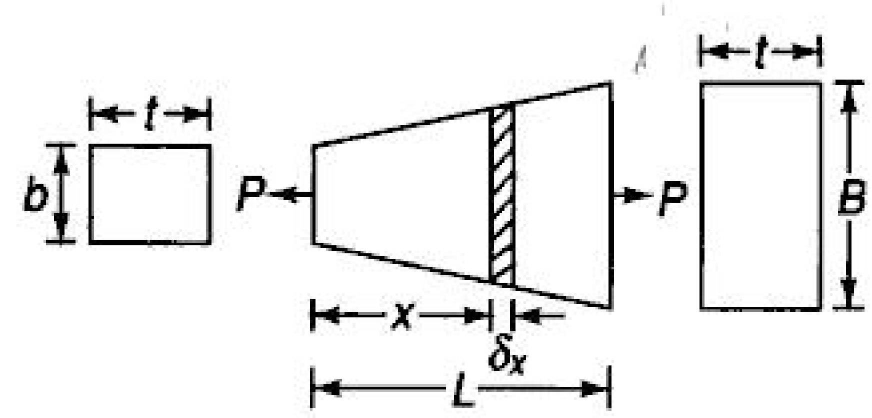


Fig. 1.9

The width at a distance x from the small end

$$= b + \frac{B-b}{L}.x = b + kx$$

....[Taking $k = (B - b)/L$]

The area of cross-section at this distance = $(b + kx).t$

The extension of the small length = $\dfrac{P.\delta x}{(b + kx)t.E}$

Extension of the whole rod

$$= \int_0^L \frac{P}{(b + kx)t.E}.dx = \frac{P}{tE}\int_0^L \frac{1}{(b + kx)}.dx$$

$$= \frac{P}{tE}\frac{1}{k}\left[\log_e(b + kx)\right]_0^L = \frac{P}{ktE}\left(\log_e \frac{b + kL}{b}\right) = \frac{P}{ktE}\log_e \frac{B}{b} \qquad (1.8)$$

$$\ldots\ldots \left(b + kL = b + \frac{B-b}{L}.L = B\right)$$

1.9 ELONGATION DUE TO SELF-WEIGHT

The elongation due to self-weight of bars of rectangular and conical sections may be considered as follows:

Rectangular Section

Consider a bar hanging freely under its own weight as shown in Fig. 1.10.

Consider a small length δx of the bar at a distance x from the free end.

Let A = area of cross-section of the bar

w = weight per unit length of the bar

W = weight of the whole bar = wL

W_x = weight of the bar below the small section = wx

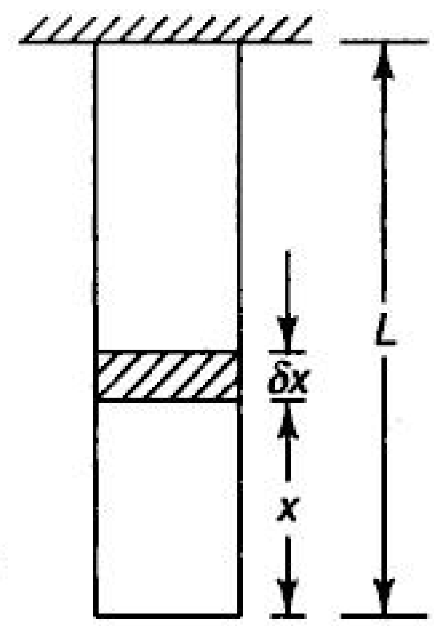


Fig. 1.10

The extension of a small length = $\frac{W_x.\delta x}{A.E} = \frac{wx.\delta x}{A.E}$

Extension of the whole rod

$$= \int_0^L \frac{wx}{AE}.dx = \frac{w}{AE}\left(\frac{x^2}{2}\right)_0^L = \frac{wL^2}{2AE} = \frac{wL.L}{2AE} = \frac{WL}{2AE} \quad (1.9)$$

= deformation due to a weight W at the lower end/2

Thus the deformation of the bar under its own weight is equal to half the deformation due to a direct load equal to the weight of the body applied at the lower end.

Conical Section

Consider a small length δx of the bar at a distance x from the free end (Fig. 1.11).

Let A = area of cross-section at the small length

w = weight per unit volume of the bar

W_x = weight of the bar below the section = $wAx/3$

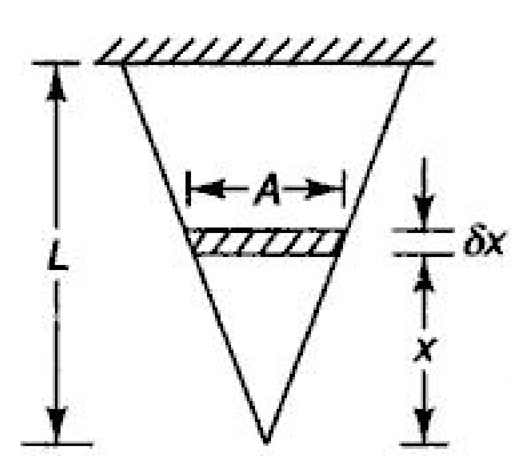


Fig. 1.11

The extension of a small length = $\frac{W_x.\delta x}{A.E} = \frac{wAx.\delta x}{3A.E}$

Extension of the whole rod = $\int_0^L \frac{wAx}{3AE}.dx = \frac{w}{3E}\int_0^L x.dx$

$$= \frac{wL^2}{6E} \quad (1.10)$$

Comparing it with Eq. 1.9, this elongation is one-third that of the rectangular section of the same length under own weight of the bar.

1.10 COLUMN OF UNIFORM STRENGTH

Let a bar of varying cross-sectional area be acted upon by a load P as shown in Fig. 1.12. Consider a small length dx at a distance x from the top.

Let A = area at distance x

$A + dA$ = area at distance $x + dx$

w = weight per unit volume of the bar

Considering the balance of forces acting on the small length,

$\sigma(A + dA) = \sigma A$ + weight of the small length dx of the bar

or $\sigma(A + dA) = \sigma A + wAdx$

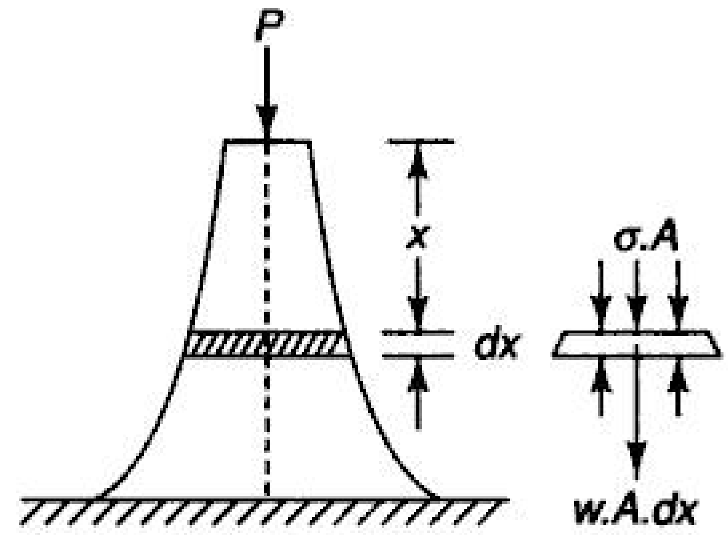


Fig. 1.12

or $\sigma.dA = wAdx$ or $\dfrac{dA}{A} = \dfrac{w}{\sigma}dx$

Integrating both sides, $\log_e A = \dfrac{w}{\sigma}x + C$

At the top, where $x = 0$, let Area $A = a$

Then, $\log_e a = 0 + C$ or $C = \log_e a$

Thus $\log_e A = \dfrac{w}{\sigma} + \log_e a$ or $\log_e \dfrac{A}{a} = \dfrac{w}{\sigma}x$

or $\dfrac{A}{a}\; e^{wx/\sigma}$

or $A = ae^{wx/\sigma}$ (1.11)

1.11 STATICALLY INDETERMINATE SYSTEMS

When a system comprises two or more members of different materials, the forces in various members cannot be determined by the principle of statics alone. Such systems are known as *statically indeterminate systems*. In such systems, additional equations are required to supplement the equations of statics to determine the unknown forces. Usually, these equations are obtained from deformation conditions of the system and are known as *compatibility equations*. A compound bar is a case of an indeterminate system and is discussed below:

Compound Bar

A bar consisting of two or more bars of different materials in parallel is known as a *composite* or *compound bar*. In such a bar, the sharing of load by each can be found by applying equilibrium and the compatibility equations.

Consider the case of a solid bar enclosed in a hollow tube as shown in Fig. 1.13. Let the subscripts 1 and 2 denote the solid bar and the hollow tube respectively.

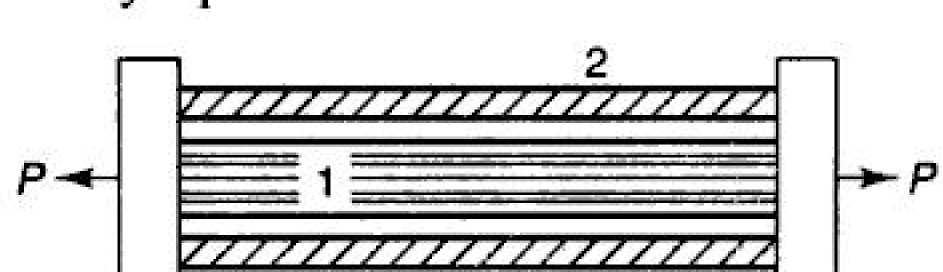


Fig. 1.13

Equilibrium equation As the total load must be equal to the load taken by individual members,

$$P = P_1 + P_2 \tag{i}$$

Compatibility equation The deformation of the bar must be equal to the tube.

$$\frac{P_1L}{A_1E_1} = \frac{P_2L}{A_2E_2} \quad \text{or} \quad P_1 = \frac{P_2A_1E_1}{A_2E_2} \tag{ii}$$

Inserting (ii) in (i),

$$P = \frac{P_2A_1E_1}{A_2E_2} + P_2 = \frac{P_2A_1E_1 + P_2A_2E_2}{A_2E_2} = \frac{P_2(A_1E_1 + A_2E_2)}{A_2E_2}$$

or
$$P_2 = \frac{P.A_2E_2}{A_1E_1 + A_2E_2} \tag{1.12}$$

Similarly,
$$P_1 = \frac{P.A_1E_1}{A_1E_1 + A_2E_2} \tag{1.13}$$

Example 1.3 *Three equally spaced rods in the same vertical plane support a rigid bar AB. Two outer rods are of brass, each 600-mm long and of 25-mm diameter. The central steel rod is 800-mm long and 30 mm in diameter. Determine the forces in the bars due to an applied load of 120 kN through the mid-point of the bar. The bar remains horizontal after the application of load. Take $E_s/E_b = 2$.*

Solution Refer Fig. 1.14.

As the bar remains horizontal after the application of load, the elongation of each of the brass bars and of the steel bar are the same.

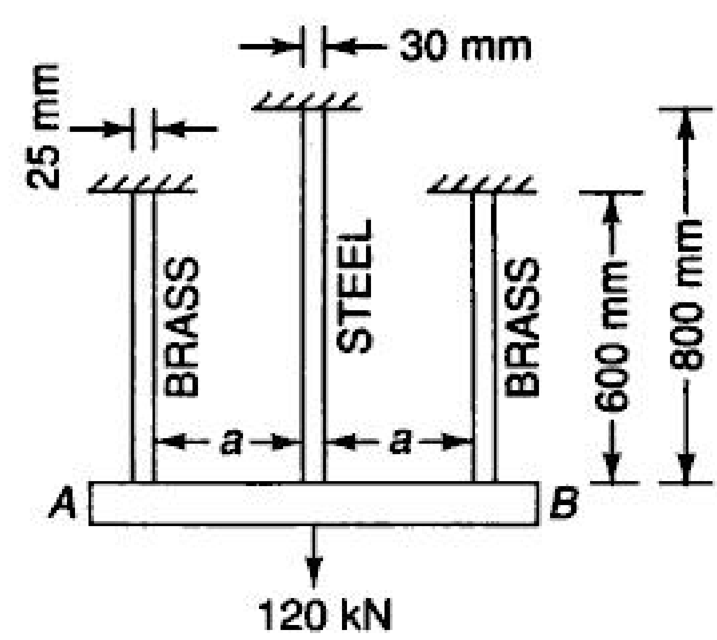


Fig. 1.14

From compatibility equation, $\Delta_b = \Delta_s$

or
$$\frac{P_bL_b}{A_bE_b} = \frac{P_sL_s}{A_sE_s}$$

or
$$P_b = \frac{L_s}{L_b}\cdot\frac{E_b}{E_s}\left(\frac{d_b}{d_s}\right)^2 P_s$$

$$= \frac{800}{600}\cdot\frac{1}{2}\left(\frac{25}{30}\right)^2 P_s$$

or $P_b = 0.463\ P_s$

From equilibrium equation, $2P_b + P_s = P$

or $2 \times 0.463\ P_s + P_s = 120$ or $1.926\ P_s = 120$

or $P_s = 62.3$ kN and $P_b = 28.84$ kN

Example 1.4 *Three equidistant vertical rods each of 20-mm diameter support a load of 25 kN in the same plane as shown in Fig. 1.15. Initially, all the rods are adjusted to share the load equally. Neglecting any chance of buckling, and taking $E_s = 205$ GPa and $E_c = 100$ GPa, determine the final stresses when a further load of 20 kN is added.*

Solution $A = (\pi/4)\, 20^2 = 100\,\pi\ \text{mm}^2$

Initially, the stress in each rod $= \dfrac{25\,000}{100\pi \times 3} = 26.53$ MPa

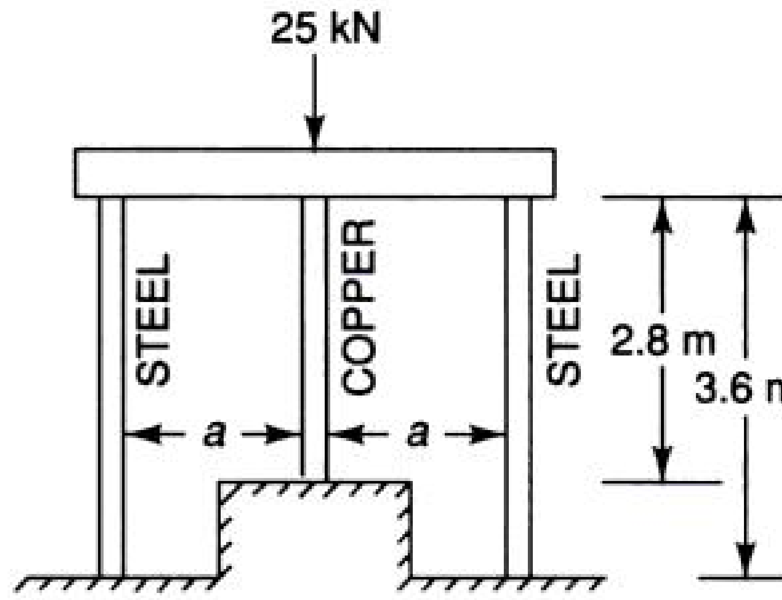


Fig. 1.15

On adding a further load of 20 kN, let the increase of stress in the steel rod be σ_s and in the copper rod σ_c.

Then from equilibrium equation, the additional load P is

$$(2\sigma_s + \sigma_c)A = P \text{ or } (2\sigma_s + \sigma_c) \times 100\pi = 20\,000 \quad \text{(i)}$$

From compatibility equation, $\Delta_c = \Delta_s$

$$\frac{\sigma_c L_c}{E_c} = \frac{\sigma_s L_s}{E_s}$$

or $\sigma_c = \dfrac{L_s}{L_c} \cdot \dfrac{E_c}{E_s} \sigma_s = \dfrac{3.6}{2.8} \times \dfrac{100\,000}{205\,000} \sigma_s$ or $\sigma_c = 0.627\sigma_s$

Inserting this value of σ_c in (i)

$\therefore \quad (2\sigma_s + 0.627\,\sigma_s) \times 100\pi = 20\,000$

or $\quad 2.627\,\sigma_s = 63.662$

or $\quad \sigma_s = 24.23$ MPa and $\sigma_c = 15.19$ MPa

Final stress in steel rod = 24.23 + 26.53 = 50.76 MPa

Final stress in copper rod = 15.19 + 26.53 = 41.72 MPa

Example 1.5 *A steel rod of 16-mm diameter passes through a copper tube of 20 mm internal diameter and of 32-mm external diameter. The steel rod is fitted with nuts and washers at each end. The nuts are tightened till a stress of 24 MPa is developed in the steel rod. A cut is then made in the copper tube for half the length to remove 2 mm from its thickness. Assuming the Young's modulus of steel to be twice that of copper, determine*

(i) the stress existing in the steel rod.

(ii) the stress in the steel rod if a compressive load of 4 kN is applied to the ends of the steel rod.

Solution Refer Fig. 1.16.

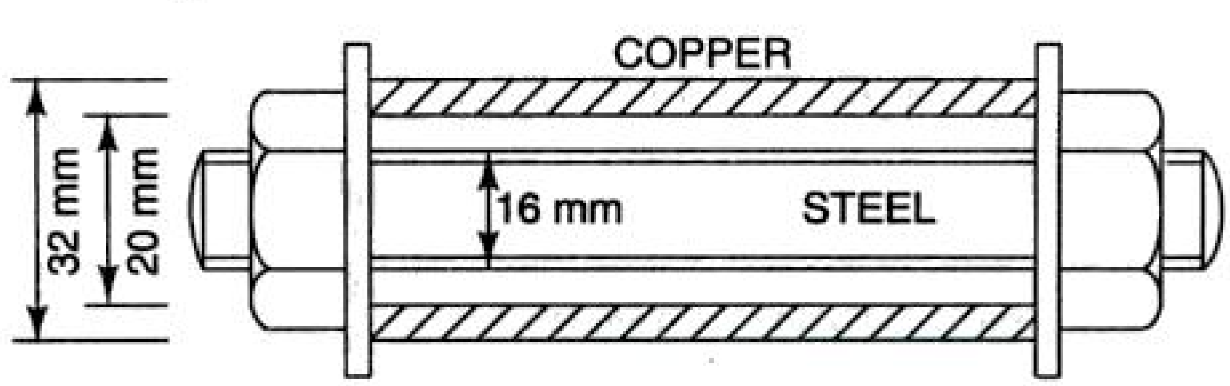


Fig. 1.16

$$A_s = \frac{\pi}{4} \times 16^2 = 64\pi \text{ mm}^2$$

and $$A_c = \frac{\pi}{4}(32^2 - 20^2) = 156\pi \text{ mm}$$

On tightening the nut, the steel rod is elongated and the stress induced is tensile whereas the tube is shortened and the stress is compressive.

Let σ_{s1} = stress in the steel rod = 24 MPa

σ_{c1} = stress in the copper tube

From equilibrium equation

Push on copper tube = Pull on steel rod

$$\sigma_{c1} \times A_c = \sigma_{s1} \times A_s \quad \text{or} \quad \sigma_{c1} \times 156\pi = 24 \times 64\pi$$

or $\sigma_{c1} = 9.846$ MPa (compressive)

(i) When the copper tube is reduced in diameter,

$A_{c'}$ = reduced area of cross-section of the tube = $\frac{\pi}{4}(28^2 - 20^2) = 96\pi$

Let σ_{s2} = stress in the steel rod

$\sigma_{c2'}$ = stress in the reduced section of tube

and σ_{c2} = stress in the remaining section of tube

From equilibrium equation,

Force in each section of copper tube as well as in the steel rod are to be equal

i.e. $$\sigma_{c2} \times 156\pi = \sigma_{c2'} \times 96\pi = \sigma_{s2} \times 64\pi \qquad \text{(i)}$$

$$\sigma_{c2} = 0.4103\ \sigma_{s2} \text{ and } \sigma_{c2'} = 0.6667\ \sigma_{s2}$$

From compatibility equation,

When the cross-section of the tube is reduced, the change in length of the rod as well as of the tube is to of the same nature, i.e. either the length of both is increased or decreased. Let us assume that the length of each is reduced which means a reduction of tensile stress in the rod and increase of compressive stress in the tube.

Thus reduction in length of steel rod = reduction in length of copper tube

$$\frac{\sigma_{s1} - \sigma_{s2}}{E_s}.L = \frac{\sigma_{c2} - \sigma_{c1}}{E_c}.\frac{L}{2} + \frac{\sigma'_{c_2} - \sigma_{c1}}{E_c}.\frac{L}{2}$$

or $$\sigma_{s1} - \sigma_{s2} = \sigma_{c2} + \sigma'_{c2} - 2\sigma_{c1} \quad \ldots\ldots \quad (E_s = 2E_c) \qquad \text{(ii)}$$

or $$24 - \sigma_{s2} = 0.4103\ \sigma_{s2} + 0.6667\sigma_{s2} - 2 \times 9.846$$

or $$2.077\sigma_{s2} = 43.692 \quad \text{or} \quad \sigma_{s2} = 21.036 \text{ MPa}$$

As the stress in the steel rod is decreased from 24 MPa to 21.036 MPa, the assumption of reduction of the length of the two is correct. In case, the lengths are assumed to be increased, the stress in the steel rod is increased and in the copper tube decreased. The equation formed would have been

$$\frac{\sigma_{s2} - \sigma_{s1}}{E_s}.L = \frac{\sigma_{c1} - \sigma_{c2}}{E_s}.\frac{L}{2} + \frac{\sigma_{c1} - \sigma'_{c2}}{E_s}.\frac{L}{2}$$

and the result would have been the same i.e. σ_{s2} = 21.036 MPa which would have indicated that the length actually would be reduced due to decrease in the stress of steel rod.

(ii) When a compressive load of 4 kN is applied to the ends of the steel rod, the length of the rod is further reduced.

Equilibrium equation

$$\sigma_{c3} \times 156\pi = \sigma'_{c3} \times 96\pi = \sigma_{s3} \times 64\pi + 6000 \qquad \text{[as in (i)]}$$

or $$\sigma_{c3} \times 156 = \sigma'_{c3} \times 96 = \sigma_{s3} \times 64 + 1909.9$$

or $$\sigma_{c3} = 0.4103\ \sigma_{s3} + 12.243$$

and $$\sigma'_{c3} = 0.6667\ \sigma_{s3} + 19.895$$

Compatibility equation

$$\sigma_{s1} - \sigma_{s3} = \sigma_{c3} + \sigma'_{c3} - 2\sigma_{c1} \qquad \text{[as in (ii)]}$$

or $$24 - \sigma_{s3} = 0.4103\sigma_{s3} + 12.243 + 0.6667\sigma_{s3} + 19.895 - 2 \times 9.846$$

or $$2.077\sigma_{s3} = 11.554 \quad \text{or} \quad \sigma_{s3} = 5.56 \text{ MPa}$$

Example 1.6 *A round steel rod supported in a recess is surrounded by a co-axial brass tube as shown in Fig. 1.17. The level of the upper end of the rod is 0.08 mm below that of the tube. Determine:*

(i) the magnitude and direction of the maximum permissible axial load which can be applied to a rigid plate resting on the top of the tube. The permissible values of the compressive stresses are 105 MPa for steel and 75 MPa for brass.

(ii) the amount by which the tube is shortened by a load if the compressive stresses in the steel and the brass are the same.

Take E_s = 210 GPa and E_b = 105 GPa.

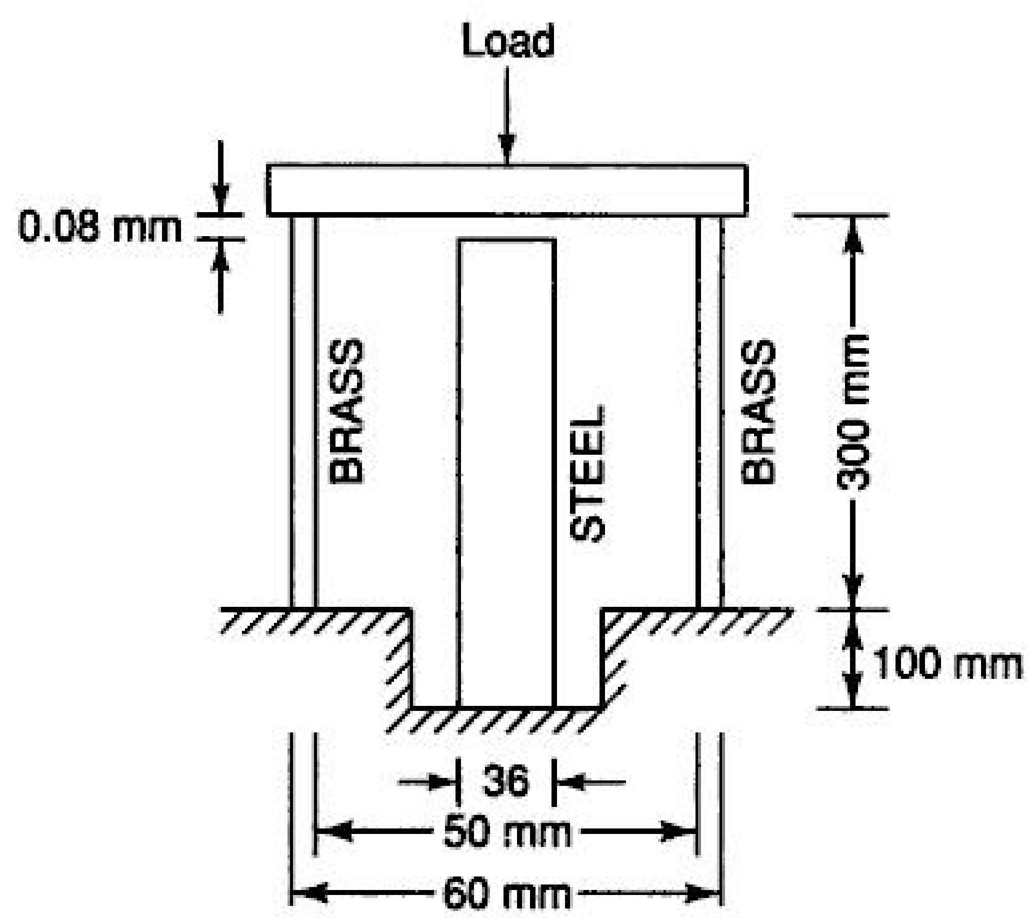


Fig. 1.17

Solution $$A_s = \frac{\pi}{4} \times 36^2 = 324\pi \text{ mm}^2$$

and $$A_b = \frac{\pi}{4} \times (60^2 - 50^2) = 275\pi \text{ mm}^2$$

(i) Let W_b be the load applied for the initial compression of the tube before the compression of the rod starts. Then

$$\Delta_b = \frac{\sigma_b L}{E} \quad \text{or} \quad 0.08 = \frac{\sigma_b \times 300}{105\,000}$$

or $$\sigma_b = 28 \text{ MPa and } W_b = 28 \times 275\pi = 24\,190 \text{ N}$$

But limiting value of stress in the brass = 75 MPa

∴ Maximum value of stress due to additional load can be = 75 – 28 = 47 MPa

Let W be the additional load to compress both, the tube and the bar. Let σ_s be the stress induced in the steel rod and σ_b the additional stress in the brass tube.

Equilibrium equation, $\sigma_s A_s + \sigma_b A_b = W$

Compatibility equation, $\Delta_s = \Delta_b$

or $\dfrac{\sigma_s L_s}{E_s} = \dfrac{\sigma_b L_b}{E_b}$ or $\sigma_s = \dfrac{L_b}{L_s} \cdot \dfrac{E_s}{E_b} \sigma_b = \dfrac{300}{400} \cdot \dfrac{210}{105} \sigma_b$ or $\sigma_s = 1.5\sigma_b$

Therefore the stress induced in the steel rod = 1.5 × 47 = 70.5 MPa

It is less than the permissible value of stress for steel.

Thus $\quad W = p_s A_s + p_b A_b = 70.5 \times 324\pi + 47 \times 275\pi = 112\ 365$ N

Total maximum load = 112 365 + 24 190 = 136 555 N or 136.555 kN

(ii) Let Δ be the shortening of the steel rod. This will also be the additional shortening of the brass tube. Then $\Delta_b + \Delta = \dfrac{\sigma_b . L_b}{E_b}$

or $\quad \sigma_b = \dfrac{105\ 000}{300}(0.08 + \Delta)$ and $\sigma_s = \dfrac{210\ 000}{400}.\Delta$

Equating the stresses in the steel and the brass,

$$= \frac{105\ 000}{300}(0.08 + \Delta) = \frac{210\ 000}{400}\Delta \text{ or } 0.08 + \Delta = 1.5\ \Delta$$

or $\quad 0.5\ \Delta = 0.08$ or $\Delta = 0.16$ mm

Total shortening = 0.08 + 0.16 = 0.24 mm

Example 1.7 *Three wires of the same material and cross-section support a rigid bar which further supports a weight of 5 kN. The length of the wires is 5 m, 8 m and 6 m in order. The spacing between the wires is 2 m and the weight acts at 1.6 m from the first wire. Determine the load carried by each wire.*

Solution As the wires are of different lengths and the weight suspended is unsymmetrical, the bar will not remain horizontal but will be deformed as shown in Fig. 1.18.

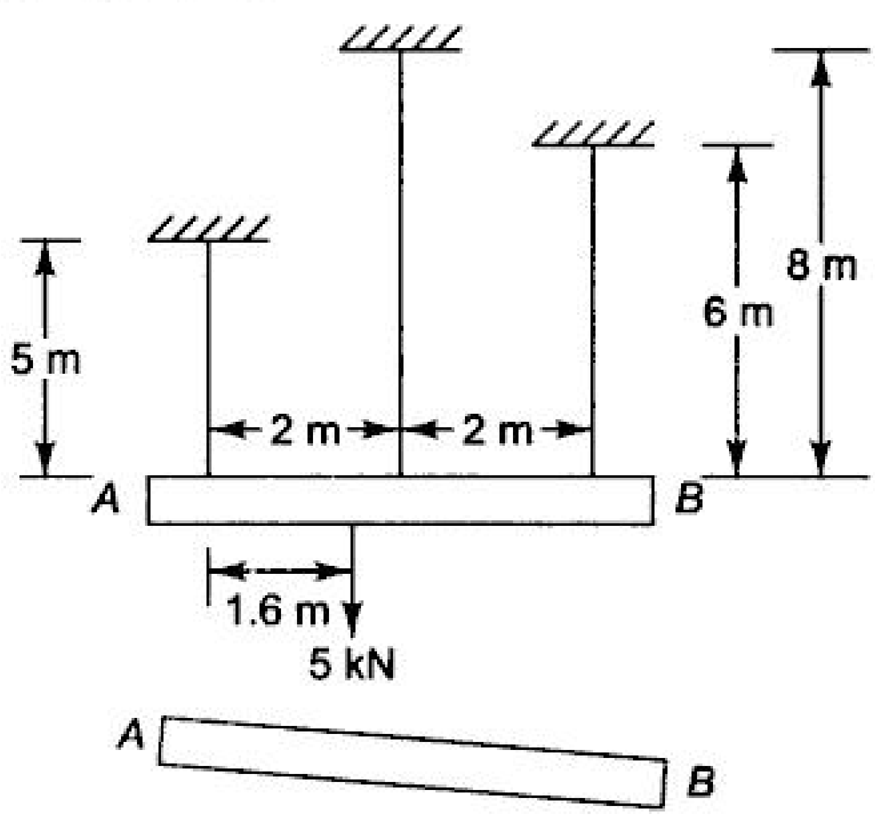


Fig. 1.18

Let P_1, P_2 and P_3 be the loads taken by the first, second and the third wire respectively.

Then $\quad P_1 + P_2 + P_3 = P = 5000 \quad$ (i)

Taking moments about the first wire,

$2P_2 + 4P_3 = 1.6 \times 5000 = 8000$

or $\quad P_2 = 4000 - 2P_3 \quad$ (ii)

Also, from symmetry,

$$\Delta_2 = \frac{\Delta_1 + \Delta_3}{2} \text{ or } 2\left(\frac{P_2 L_2}{AE}\right) = \frac{P_1 L_1}{AE} + \frac{P_3 L_3}{AE}$$

or $\quad 2P_2 L_2 = P_1 L_1 + P_3 L_3$ or $2P_2 \times 8 = P_1 \times 5 + P_3 \times 6$

or $\qquad 16P_2 = 5P_1 + 6P_3$

or $\quad 16\,(4000 - 2P_3) = 5P_1 + 6P_3$ or $64\,000 - 32P_3 = 5P_1 + 6P_3$

or $\qquad 5P_1 = 64\,000 - 38P_3$ or $P_1 = 12\,800 - 7.6P_3$ (iii)

Inserting the values of P_1 and P_2 from (ii) and (iii) in (i),

$$12\,800 - 7.6P_3 + 4000 - 2P_3 + P_3 = 5000$$

or $\qquad 8.6P_3 = 11\,800$ or $P_3 = 1372$ N or 1.372 kN

$$P_2 = 4000 - 2P_3 = 4000 - 2 \times 1372 = 1256 \text{ N or } 1.256 \text{ kN}$$

$$P_1 = 12\,800 - 7.6 \times 1372 = 2373 \text{ N or } 2.373 \text{ kN}$$

Example 1.8 *A system of three bars supports a vertical load P as shown in Fig. 1.19. The outer bars are identical and of the same material whereas the inner bar is of different material. Determine the forces in the bars of the system.*

Solution Owing to symmetry, forces in the outer bars 1 and 3 will be equal. Let it be P_1 and the force in the inner bar P_2. The dotted lines show the deformed shape of the system under the load P.

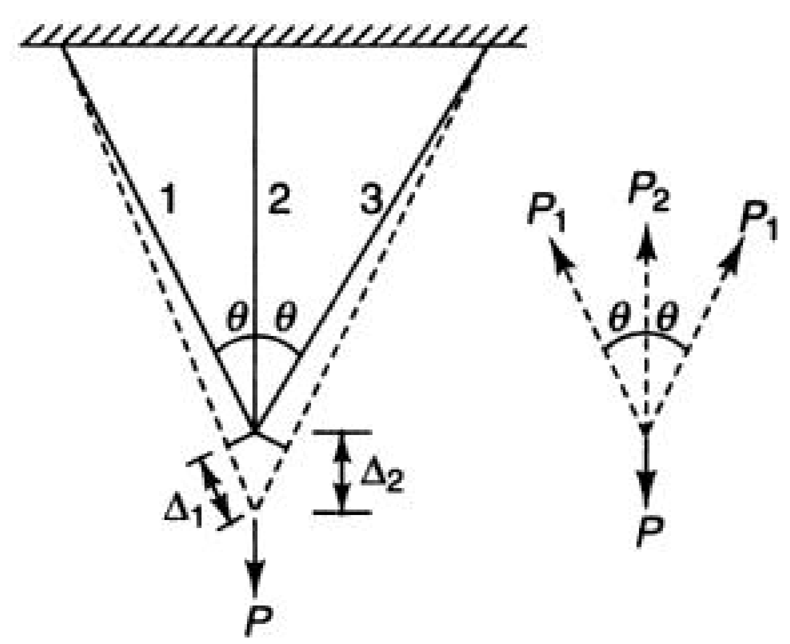


Fig. 1.19

From equilibrium equation,

$$2P_1 \cos\theta + P_2 = P \qquad \text{....(assuming negligible change in } \theta) \qquad \text{(i)}$$

From compatibility equation,

$$\Delta_1 = \Delta_2 \cos\theta \quad \text{or} \quad \frac{P_1L_1}{A_1E_1} = \frac{P_2L_2}{A_2E_2}\cos\theta$$

or $$P_1 = \frac{A_1E_1P_2L_2}{A_2E_2L_1}\cos\theta = \frac{A_1E_1P_2(L_1\cos\theta)}{A_2E_2L_1}\cos\theta = \frac{A_1E_1P_2}{A_2E_2}\cos^2\theta \qquad \text{(ii)}$$

Substituting this value of P_1 in (i),

$$2\frac{A_1E_1P_2}{A_2E_2}\cos^3\theta + P_2 = P \quad \text{or} \quad P_2 = \frac{P}{1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta}$$

From (ii), $$P_1 = \frac{A_1E_1}{A_2E_2}\cdot\frac{P}{1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta}.\cos^2\theta$$

$$= P\cos^2\theta\,\frac{1}{\left(\dfrac{A_2E_2}{A_1E_1}\right)}\cdot\frac{1}{\left(1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta\right)} = \frac{P\cos^2\theta}{\left(\dfrac{A_2E_2}{A_1E_1} + 2\cos^3\theta\right)}$$

Example 1.9 *Figure 1.20 shows a horizontal bar supported by two suspended vertical wires fixed to a rigid support. A load W is attached to the bar. The left hand side wire is of copper with a diameter of 5 mm and the right hand side wire is of steel of 3 mm diameter. The length of both the wires is 4 m initially. Find the position of the weight on the bar so that both the wires extend by the same amount.*

Also, calculate the load, stresses and the elongation of each wire if W = 1000 N. Neglect the weight of the bar and take $E_s = 210$ *GPa and* $E_c = 120$ *GPa.*

Solution $A_c = \frac{\pi}{4}(5)^2 = 6.25\pi$ mm²

and $A_s = \frac{\pi}{4}(3)^2 = 2.25\pi$ mm²

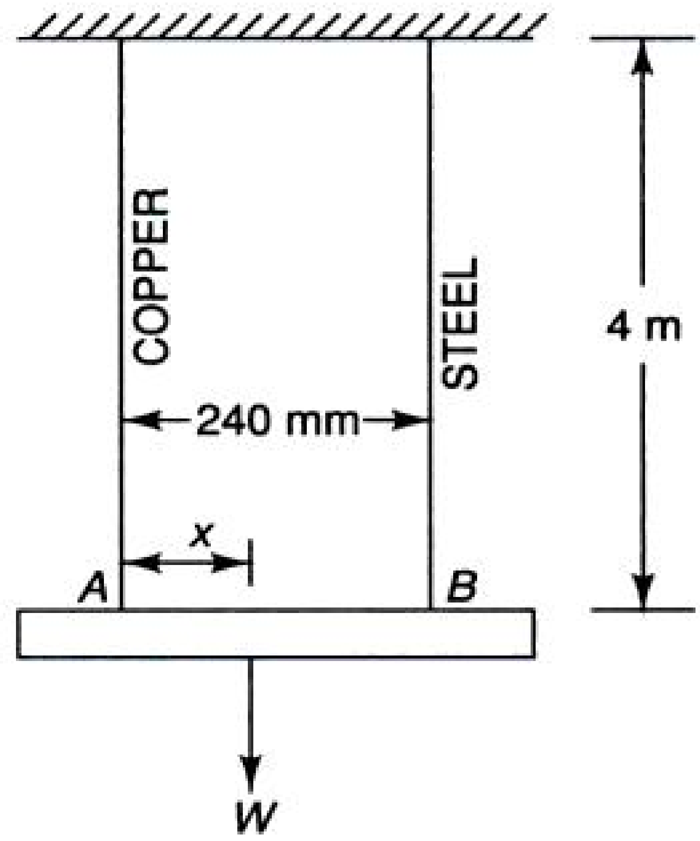


Fig. 1.20

Let the load W be placed at a distance x from the copper wire and P_s and P_c the forces in steel and copper wires respectively.

Then taking moments about A, $240\ P_s = W.x$

or $P_s = \dfrac{W.x}{240}$ (i)

Taking moments about B, $240\ P_c = W(240 - x)$

or $P_c = \dfrac{W.(240-x)}{240}$ (ii)

Dividing (ii) by (i), $\dfrac{P_c}{p_s} = \dfrac{240-x}{x}$ *(iii)*

As both the wires extend by the same amount, $\Delta_c = \Delta_s$

or $$\frac{P_c L_c}{A_c E_c} = \frac{P_s L_s}{A_s E_s} \text{ or } \frac{P_c}{p_s} = \frac{A_c}{A_s}\cdot\frac{E_c}{E_s} \quad \ldots\ldots \quad (\because L_c = L_s)$$

$$= \frac{6.25\pi}{2.25\pi}\cdot\frac{120\,000}{210\,000} = 1.587 \quad \text{(iv)}$$

From (iii) and (iv), $\dfrac{240-x}{x} = 1.587$ or $x = 92.77$ mm

Numerical:

$$P_c = \frac{W(240-x)}{240} = \frac{1000\times(240-92.77)}{240} = 613.46 \text{ N}$$

$$P_s = \frac{Wx}{240} = \frac{1000\times 92.77}{240} = 386.54 \text{ N}$$

$$\sigma_c = \frac{P_c}{A_c} = \frac{613.46}{6.25\pi} = 31.24 \text{ MPa}$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{386.54}{2.25\,\pi} = 54.68 \text{ MPa}$$

$$\Delta = \frac{\sigma_c.L}{E_c} = \frac{31.24\times 4000}{120\,000} = 1.041 \text{ mm}$$

Example 1.10 *Three identical pin-connected bars support a load P as shown in Fig. 1.21. All the bars are of the same area of cross-section and same length. Determine*

(i) the force in each bar

(ii) the vertical displacement of the point where the load is applied

Neglect the possibility of lateral buckling of the bars.

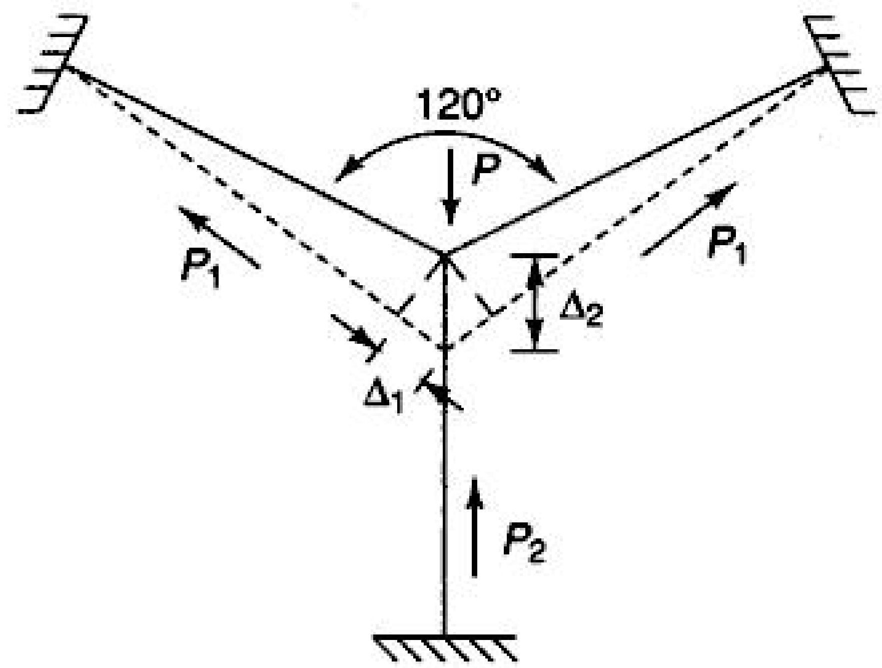


Fig. 1.21

Solution

(i) The dotted lines show the deformed shape of the structure. Assuming that there is negligible change in the angles after the deforming of the bars.

Equilibrium equation

$$2P_1 \cos 60^\circ + P_2 = P$$

or $$P_1 = P - P_2 \quad \text{(i)}$$

Compatibility equation, $\Delta_1 = \Delta_2 \cos 60^\circ$

or $$\frac{P_1 L}{AE} = \frac{P_2 L}{AE} \cos 60^\circ \text{ or } P_1 = \frac{P_2}{2} \quad \text{(ii)}$$

From (i) and (ii), $\frac{P_2}{2} = P - P_2$ or $P_2 = \frac{2P}{3}$ and $P_1 = P/3$

(ii) Vertical displacement of the joint, $\Delta_2 = \frac{P_2 L}{AE} = \frac{2PL}{3AE}$

Example 1.11 *A bar system is loaded as shown in Fig. 1.22. Determine*

(i) the reaction of the lower support, and (ii) the stresses in the bars.

Take E = 205 GPa

Solution

(i) When the load is applied and the support touches it, the reactions of both the supports will be upward since the load is downward.

Let R_1 = reaction of the upper support

R_2 = reaction of the lower support

Then $R_1 + R_2 = 40\ 000$

or $R_1 = 40\ 000 - R_2$

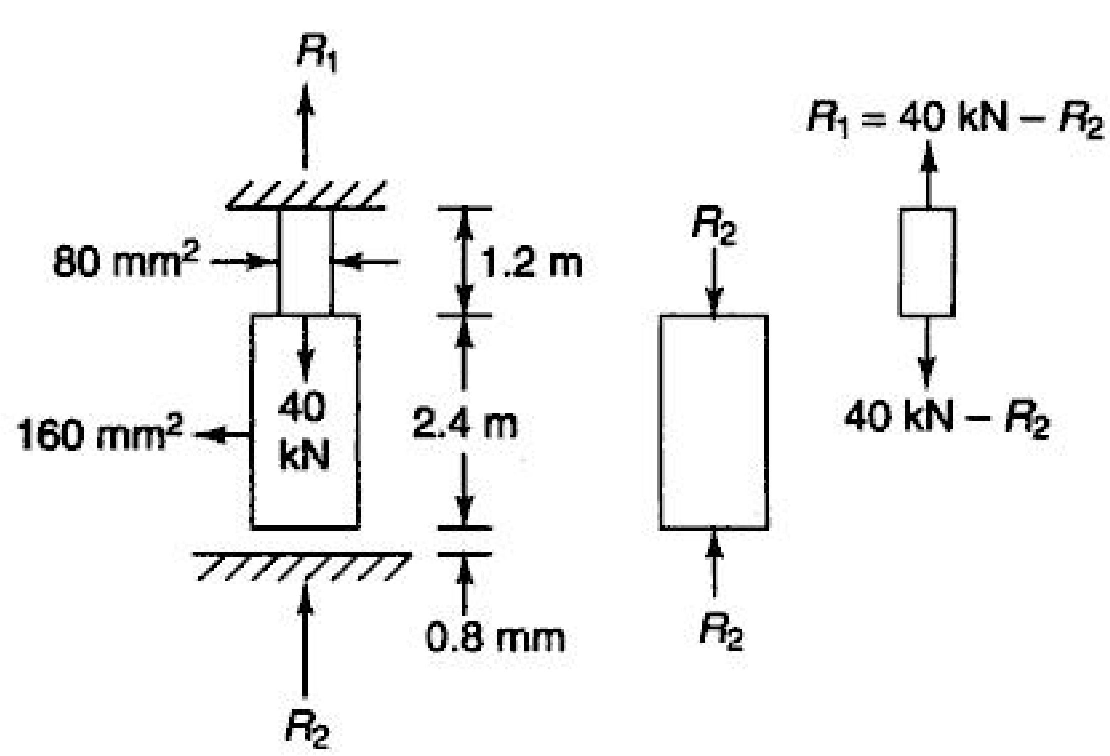


Fig. 1.22

The free-body diagrams of the two portions of the bar system is shown in the figure. It is clear that the upper portion is in tension whereas the lower portion in compression.

Elongation of the upper portion, $\Delta_1 = \frac{P_1 L_1}{A_1 E} = \frac{(40\,000 - R_2) \times 1200}{80 \times 205\,000}$

Shortening of the lower portion, $\Delta_2 = \frac{P_2 L_2}{A_2 E} = \frac{R_2 \times 2400}{160 \times 205\,000}$

From compatibility equation,

Elongation of upper portion – shortening of lower portion = net elongation = 0.8 mm

or $$\frac{(40\,000 - R_2) \times 1200}{80 \times 205\,000} - \frac{R_2 \times 2400}{160 \times 205\,000} = 0.8$$

$$(40\,000 - R_2) \times 15 - 15R_2 = 0.8 \times 205\,000$$

or $40\,000 - 2R_2 = 10\,933$ or $R_2 = 14\,533$ N

and $R_1 = 40\,000 - 14\,533 = 25\,467$ N

(ii) $\sigma_1 = 25\,467/80 = 318.3$ MPa (tensile)

$\sigma_2 = 14\,533/160 = 90.8$ MPa (compressive)

Example 1.12 *A rigid horizontal bar AB hinged at A is supported by a 1.2-m long steel rod and a 2.4-m long bronze rod, both rigidly fixed at the upper ends (Fig. 1.23). A load of 48 kN is applied at a point 3.2 m from the hinge point A. The areas of cross-section of the steel and bronze rods are 850 mm² and 650 mm² respectively. Find*

(i) stress in each rod (ii) reaction at the pivot point.

E_s = 205 GPa and E_b = 82 GPa

Solution Refer Fig. 1.23.

(i) Let P_s and P_b be the forces in the steel and bronze wires respectively as the load is applied. Taking moments about the pivot point,

$P_s \times 800 + P_b \times 2400 - 48\,000 \times 3200 = 0$

$P_s + 3P_b = 192\,000$ (i)

From compatibility equation,

$$\frac{\Delta_s}{\Delta_b} = \frac{800}{2400} = \frac{1}{3} \text{ or } \frac{\Delta_s}{1} = \frac{\Delta_b}{3}$$

or $$\frac{P_s L_s}{A_s E_s} = \frac{1}{3} \frac{P_b L_b}{A_b E_b}$$

or $$\frac{P_s \times 1200}{850 \times 205\,000} = \frac{1}{3} \times \frac{P_b \times 2400}{650 \times 82\,000}$$

$$P_s = 2.179\, P_b \quad (ii)$$

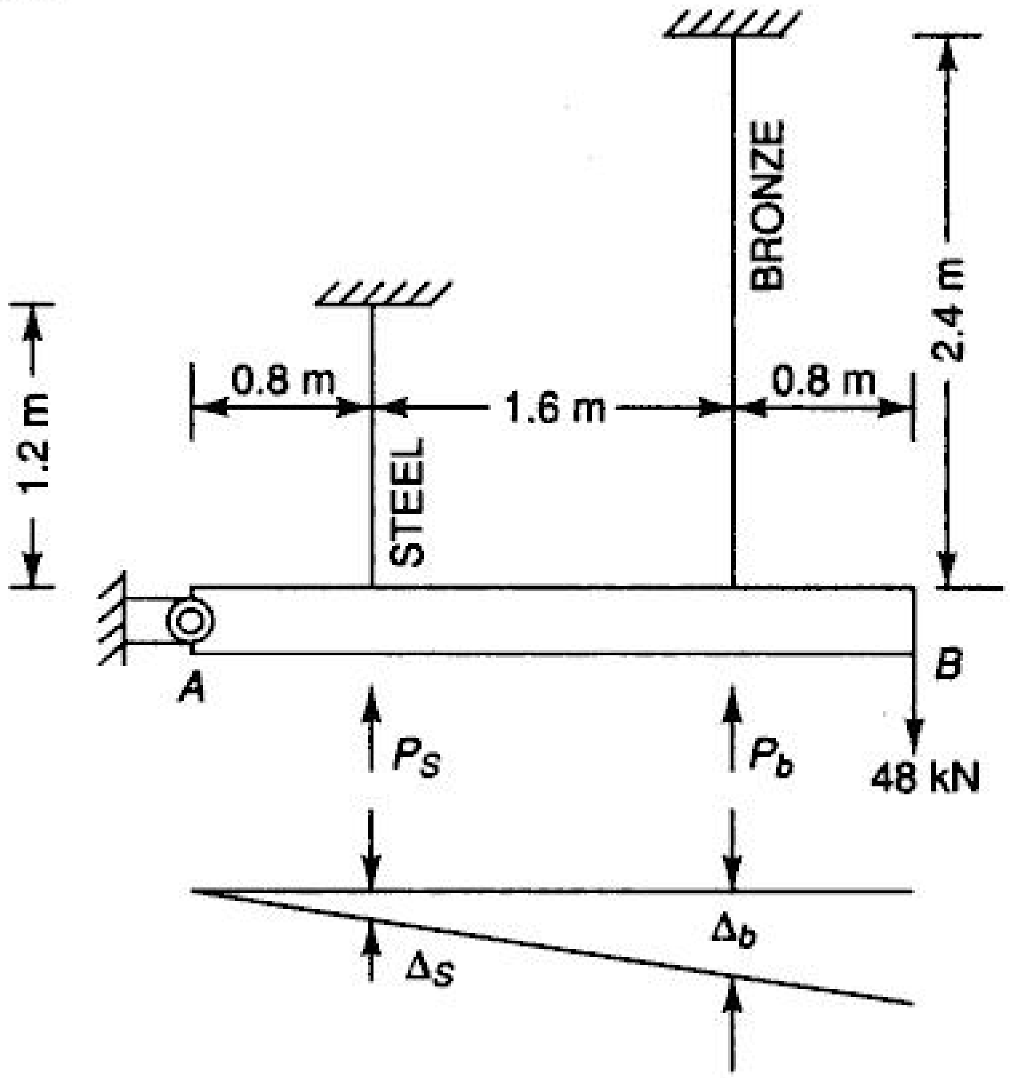


Fig. 1.23

From (i) and (ii), $2.179\, P_b + 3P_b = 192\,000$

or $P_b = 37\,073$ N

$$P_s = 192\,000 - 3 \times 37\,073 = 80\,781 \text{ N}$$

$$\sigma_b = \frac{37\,073}{650} = 57.04 \text{ MPa} \text{ and } \sigma_s = \frac{80\,781}{850} = 95.04 \text{ MPa}$$

(ii) The reaction at the pivot can be found from force equation, let it be downwards,

$$P_s + P_b - R_a = 48\,000$$

$$R_a = 80\,781 + 37\,073 - 48\,000 = 69\,854 \text{ N or } 69.854 \text{ kN}$$

Thus the assumed direction is correct.

Example 1.13 *A rigid bar AB is to be suspended from three steel rods as shown in Fig. 1.24a. The lengths of the outer rods are 1.5 m each whereas the length of the middle rod is shortened than these by an amount of 0.8 mm. The area of cross-section of all the rods is the same and is equal to 1600 mm². Determine the stresses in the rods after the assembly of the structure.* E_s *= 205 GPa.*

Solution The position of the rigid bar after the assembly is shown in Fig. 1.24*b*. It is raised upward by amounts Δ_1, Δ_2 and Δ_3 at the rod positions 1, 2 and 3 respectively. Thus the rods 1 and 3 are shortened by amounts Δ_1 and Δ_3 respectively whereas rod 2 is elongated by an amount $(0.8 - \Delta_2)$.

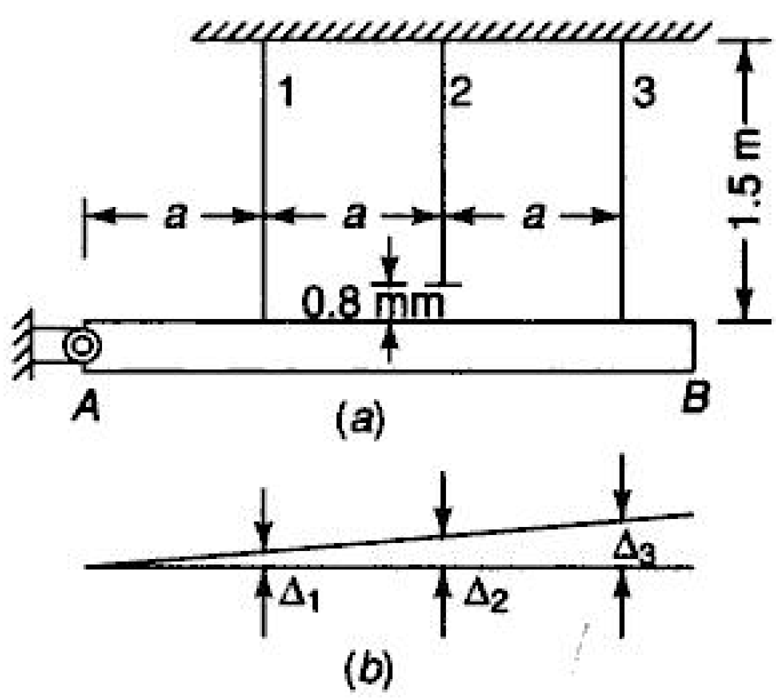


Fig. 1.24

We have, $\dfrac{\Delta_3}{\Delta_1} = \dfrac{3a}{a}$

or $\dfrac{P_3L_3/A_3E_3}{P_1L_1/A_1E_1} = 3$

As $L_3 = L_1, A_3 = A_1 \text{ and } E_3 = E_1$

$\therefore$ $P_3 = 3P_1$ (i)

Also, $\dfrac{\Delta_2}{\Delta_1} = \dfrac{2a}{a}$ or $\dfrac{0.8 - P_2L_2/A_2E_2}{P_1L_1/A_1E_1} = 2$

The length of the rod 2 is shorter by 0.8 mm. However, to find the elongation of the rod, this may be ignored as its effect will be negligible and the length of rod 2 can be taken equal to that of rod 1.

Thus $L_2 = L_1$

Also, $A_2 = A_1 \text{ and } E_2 = E_1$

$\therefore$ $\dfrac{0.8 - P_2L_1/A_1E_1}{P_1L_1/A_1E_1} = 2$ or $\dfrac{0.8 \times A_1E_1}{P_1L_1} - \dfrac{P_2}{P_1} = 2$

or $\dfrac{0.8 \times 1600 \times 205\,000}{P_1 \times 1500} - \dfrac{P_2}{P_1} = 2$ or $174\,933 - P_2 = 2P_1$

or $2P_1 + P_2 = 174\,933$ (ii)

Taking moments about A, $P_1.a + P_3.3a = P_2.2a$

or $P_1 + 3P_3 = 2P_2$ (iii)

Solving (i), (ii) and (iii),

From (i) and (iii), $P_1 + 3 \times 3P_1 = 2P_2$ or $P_2 = 5P_1$ (iv)

From (ii) and (iv), $2P_1 + 5P_1 = 174\,933$ or $P_1 = 24\,990$ N

$$P_2 = 24\,990 \times 5 = 124\,952 \text{ N}$$

$$P_3 = 24\,990 \times 3 = 74\,971 \text{ N}$$

$$\sigma_1 = 24\,990/1600 = 15.62 \text{ MPa (compressive)}$$

$$\sigma_2 = 15.62 \times 5 = 78.1 \text{ MPa (tensile)}$$

$$\sigma_3 = 15.62 \times 3 = 46.86 \text{ MPa (compressive)}$$

1.12 TEMPERATURE STRESSES

The length of a material which undergoes a change in temperature also changes and if the material is free to do so, no stresses are developed in the material. However, if the material is constrained, stresses are developed in the material which are known as *temperature stresses*.

Consider a bar of length L. If its temperature is increased through $t°$, its length is increased by an amount $L.\alpha t$, where α is the coefficient of thermal expansion. But if the bar is constrained and is prevented from expansion, the temperature stress is induced in the material which is given by

$$E = \frac{\text{temperature stress}}{\text{temperature strain}} = \frac{\sigma}{L\alpha t/L}$$

or $\sigma = \alpha t E$ (1.14)

or $\sigma = \alpha t\, \sigma/\varepsilon$

or temperature strain, $\varepsilon = \alpha t$ (1.15)

Compound Sections

Consider a copper rod enclosed in a steel tube as shown in Fig. 1.25 rigidly joined at each end. Now, if the temperature is increased by $t°$, the copper rod would tend to expand more as compared to steel tube. As the two are joined together, the copper is prevented its full expansion and is put in compression. The final position of the compound bar will be as shown in the figure.

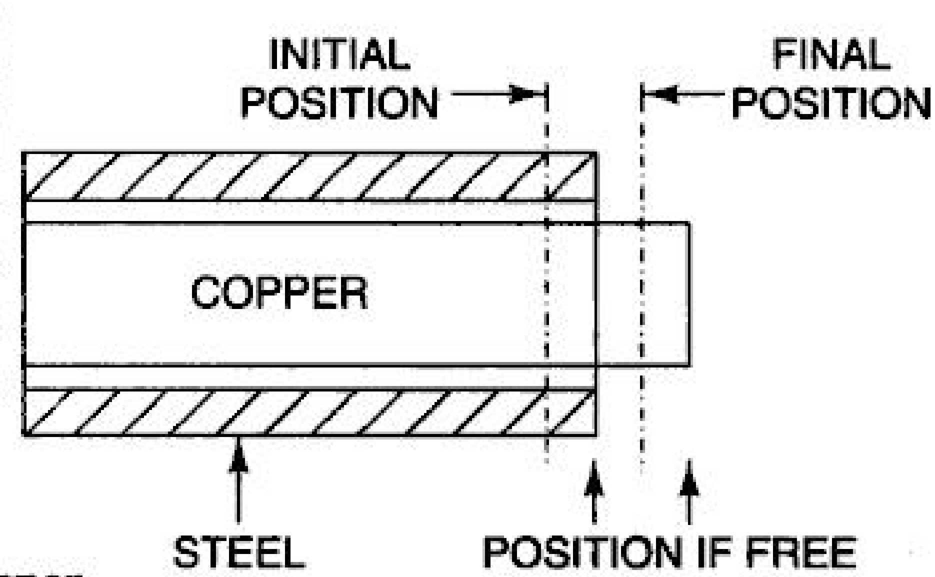


Fig. 1.25

Let σ_s = tensile stress in steel

σ_c = compressive stress in copper

A_s = cross-sectional area of steel tube

A_c = cross-sectional area of copper rod

From equilibrium equation

Tensile force in steel = compressive force in copper

$$\sigma_s . A_s = \sigma_c . A_c \quad (1.16)$$

or $$\varepsilon_s . E_s . A_s = \varepsilon_c . E_c . A_c$$

Compatibility equation:

Let α_s = coefficient of thermal expansion in steel

α_c = coefficient of thermal expansion in copper

Now Elongation of steel tube (due to temperature + due to tensile stress)
= Elongation of copper rod (due to temperature – due to compressive stress)

or Temperature strain of steel + tensile strain
= Temperature strain of copper – compressive strain

$$\alpha_s t + \sigma_s / E_s = \alpha_c t - \sigma_c / E_c$$

or $$\alpha_s t + \varepsilon_s = \alpha_c t - \varepsilon_c$$

or $$\varepsilon_s + \varepsilon_c = (\alpha_c - \alpha_s) t \quad (1.17)$$

Equations (1.16) and (1.17) are sufficient to solve the problems.

Example 1.14 *Two parallel walls 8 m apart are to be stayed together by a steel rod of 30-mm diameter with the help of washers and nuts at the ends. The steel rod is passed through the metal plates and is heated. When its temperature is raised to 90°C, the nuts are tightened. Determine the pull in the bar when it is cooled to 24°C if*

(i) the ends do not yield (ii) the total yielding at the ends is 2 mm.

E = 205 GPa and coefficient of thermal expansion of steel, $\alpha_s = 11 \times 10^{-6}$/°C.

Solution

$$A = \frac{\pi}{4}(30)^2 = 225\pi \text{ mm}^2$$

(i) Pull in the bar, $P = \sigma . A = \alpha t E A$

$$= 11 \times 10^{-6} \times (90 - 24) \times 205\,000 \times 225\,\pi = 105\,202 \text{ N}$$

(ii) When the yield at the ends is 2 mm,

$$\Delta = (\alpha L t - 2) = \frac{PL}{AE}$$

or $$P = \alpha\, t\, A\, E - \frac{2AE}{L} = 105\,202 - \frac{2 \times 225\pi \times 205\,000}{8000} = 105\,202 - 36\,227$$

$$= 68\,975 \text{ N} \quad \text{or} \quad 68.975 \text{ kN}$$

Example 1.15 *A composite bar made up of copper, steel and brass is rigidly attached to the end supports as shown in Fig. 1.26. Determine the stresses in the three portions of the bar when the temperature of the composite system is raised by 70°C when*

(i) the supports are rigid (ii) the supports yield by 0.6 mm.

$$E_c = 100 \text{ GPa}; \; E_s = 205 \text{ GPa}; \; E_b = 95 \text{ GPa}$$

$$\alpha_c = 18 \times 10^{-6}/°C; \; \alpha_s = 11 \times 10^{-6}/°C; \; \alpha_b = 19 \times 10^{-6}/°C$$

Solution

$$A_c = (\pi/4)50^2 = 625\ \pi\ \text{mm}^2;$$
$$A_s = (\pi/4)40^2 = 400\ \pi\ \text{mm}^2;\ A_b = (\pi/4)60^2 = 900\ \pi\ \text{mm}^2$$

(i) When the temperature is raised, each portion tends to elongate which is resisted by the rigid supports and the compressive stresses are developed in each portion. However, the forces so developed in each portion are equal,

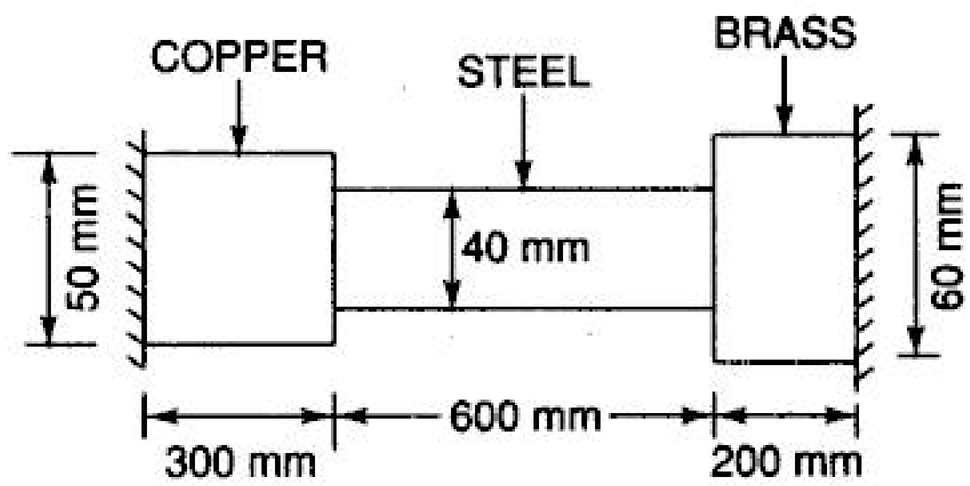


Fig. 1.26

i.e. $\sigma_c.A_c = \sigma_s.A_s = \sigma_b.A_b$ or $\sigma_c = \dfrac{A_b}{A_c}\sigma_b = \dfrac{900\pi}{625\pi} \times \sigma_b = 1.44\sigma_b$

and $\sigma_s = \dfrac{A_b}{A_s}\sigma_b = \dfrac{900\pi}{400\pi} \times \sigma_b = 2.25\sigma_b$

Elongation in the absence of supports, $\Delta = \Delta_c + \Delta_s + \Delta_b$

$$= \alpha_c L_c t_c + \alpha_s L_s t_s + \alpha_b L_b t_b$$
$$= 18 \times 10^{-6} \times 300 \times 70 + 11 \times 10^{-6} \times 600 \times 70 + 19 \times 10^{-6} \times 200 \times 70$$
$$= 70 \times 10^{-6}\ (5400 + 6600 + 3800) = 1.106\ \text{mm}$$

Also from stress considerations, $\Delta = \dfrac{\sigma_c L_c}{E_c} + \dfrac{\sigma_s L_s}{E_s} + \dfrac{\sigma_b L_b}{E_b}$

Thus, $$\frac{1.44\sigma_b \times 300}{100\ 000} + \frac{2.25\sigma_b \times 600}{205\ 000} + \frac{\sigma_b \times 200}{95\ 000} = 1.106$$

or $$(0.004\ 32 + 0.006\ 59 + 0.002\ 11)\ \sigma_b = 1.106$$

$$0.01302\ \sigma_b = 1.106$$
$$\sigma_b = 84.95\ \text{MPa}$$
$$\sigma_c = 84.95 \times 1.44 = 122.33\ \text{MPa}$$
$$\sigma_s = 84.95 \times 2.25 = 191.13\ \text{MPa}$$

(ii) When the supports yield by 0.6 mm,

$$0.0132\ \sigma_b = 1.106 - 0.6 = 0.506$$
$$\sigma_b = 38.33\ \text{MPa}$$
$$\sigma_c = 38.33 \times 1.44 = 55.20\ \text{MPa}$$
$$\sigma_s = 38.33 \times 2.25 = 86.24\ \text{MPa}$$

Example 1.16 *A steel tube of 35-mm outer diameter and 30-mm inner diameter encloses a gun metal rod of 25-mm diameter and is rigidly joined at each end. If at a temperature of 40°C there is no longitudinal stress, determine the stresses developed in the rod and the tube when the temperature of the assembly is raised to 240°C.*

Coefficient of thermal expansion of steel = 11 × 10^{-6} /°C.

Coefficient of thermal expansion of gun metal = 18 × 10⁻⁶/°C.
Young's modulus for steel = 205 GPa
Young's modulus for gun metal = 91.5 GPa
Also find the increase in length if the original length of the assembly is 1 m.

Solution Refer Fig. 1.27,

$$A_s = \frac{\pi}{4}(35^2 - 30^2) = 255.25 \text{ mm}^2$$

and $A_g = \frac{\pi}{4} \times 25^2 = 490.87 \text{ mm}^2$

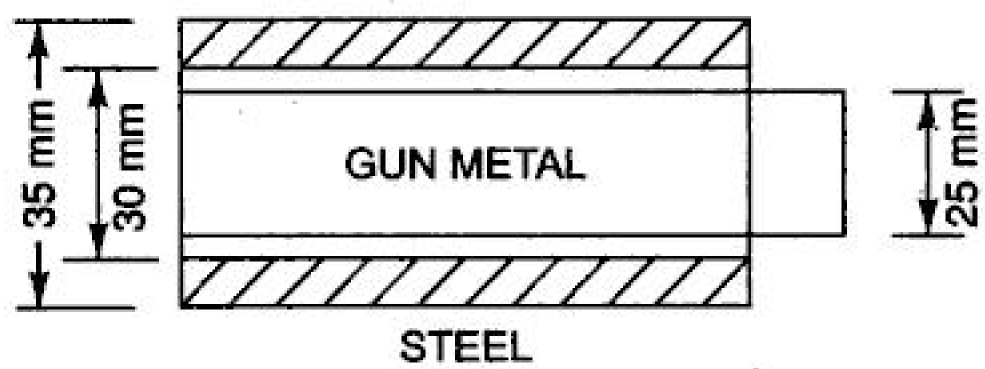


Fig. 1.27

As the coefficient of expansion of the gun metal is more as compared to that of steel, the final expansion will be less than the free expansion of gun metal due to temperature rise and thus compressive stresses will be developed in the gun metal rod. In a similar way, as the coefficient of expansion of the steel is less, the final expansion will be more than the free expansion of steel due to temperature rise and thus it will have tensile stresses.

Temperature strain of steel + tensile strain

= Temperature strain of copper – compressive strain

i.e. $$\alpha_s t + \frac{\sigma_s}{E_s} = \alpha_g t - \frac{\sigma_g}{E_g} \quad \text{or} \quad \alpha_s t + \frac{P}{A_s E_s} = \alpha_g t - \frac{P}{A_g E_g}$$

or $$P\left(\frac{1}{A_s E_s} + \frac{1}{A_g E_g}\right) = t(\alpha_g - \alpha_s)$$

or $$P = \frac{t(\alpha_g - \alpha_s)}{\frac{1}{A_s E_s} + \frac{1}{A_g E_g}} = \frac{(240-40)(18-11)\times 10^{-6}}{\frac{1}{255.25 \times 205\,000} + \frac{1}{490.87 \times 91\,500}}$$

$$= \frac{1400 \times 10^{-6}}{19.11 \times 10^{-9} + 22.26 \times 10^{-9}} = 33841 \text{ N}$$

$$\sigma_s = \frac{33841}{255.25} = 132.6 \text{ MPa and } \sigma_g = \frac{33\,841}{490.87} = 68.94 \text{ MPa}$$

Increase in length of assembly

= Elongation of steel tube (due to temperature + due to tensile stress)

= Elongation of copper rod (due to temperature – due to compressive stress)

Using the first equation, Increase in length

$$= \alpha_s L t + \frac{\sigma_s L}{E_s} = L\left(\alpha_s t + \frac{\sigma_s}{E_s}\right)$$

$$= 1000\left(11 \times 10^{-6} \times 200 + \frac{132.6}{205\,000}\right) = 2.847 \text{ mm}$$

Example 1.17 *Rails are laid such that there is no stress in them at 24° C. If the rails are 32-m long, determine*

(i) the stress in the rails at 80°C, when there is no allowance for expansion.
(ii) the stress in the rails at 80°C, when there is an expansion allowance of 8 mm per rail.
(iii) The expansion allowance for no stress in the rails at 80°C.
(iv) The maximum temperature for no stress in the rails when expansion allowance is 8 mm.

Coefficient of linear expansion, $\alpha = 11 \times 10^{-6}/°C$ *and* $E = 205$ *GPa*

Solution Change in temperature = 80° – 24° = 56°

(i) When there is no allowance for expansion,

$$\sigma = \alpha t E = 11 \times 10^{-6} \times 56 \times 205\,000 = 126.28 \text{ MPa}$$

(ii) When there is an expansion allowance of 8 mm, $\Delta = \alpha L t - 8 = \dfrac{\sigma L}{E}$

or $$11 \times 10^{-6} \times 32\,000 \times 56 - 8 = \frac{\sigma \times 32\,000}{205\,000}$$

or $19.712 - 8 = 0.1561\,\sigma$ or $\sigma = 75.03$ MPa

(iii) If stresses are to be zero, the expansion allowance

$$\Delta = \alpha L t = 11 \times 10^{-6} \times 32\,000 \times 56 = 19.71 \text{ mm}$$

(iv) For no stress in the rails when expansion allowance is 8 mm.

$$8 = \alpha L t$$

or $8 = 11 \times 10^{-6} \times 32\,000 \times t$ or $t = 22.73°C$

Example 1.18 *A steel rod of 16-mm diameter and 3-m length passes through a copper tube of 50-mm external and 40-mm internal diameter and of the same length. The tube is closed at each end with the help of 30 mm thick steel plates which are tightened by nuts till the length of the copper tube is reduced by 0.6 mm. The temperature of the whole assembly is then raised by 56°C. Determine the stresses in the steel and copper before and after the rise of temperature. Assume that the thickness of the steel plates at the ends do not change during tightening of the nuts.*

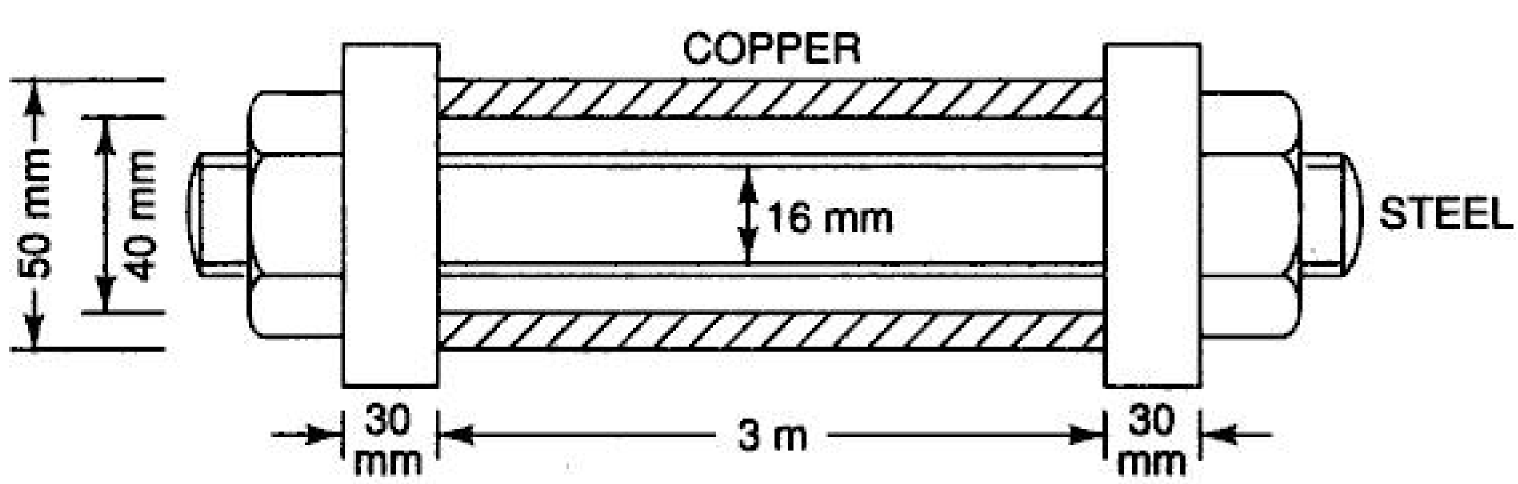


Fig. 1.28

$E_s = 210$ *GPa;* $E_c = 100$ *GPa;*

$\alpha_s = 12 \times 10^{-6}/°C;$ $\alpha_c = 17 \times 10^{-6}/°C$

Solution Refer Fig. 1.28.

$$A_s = (\pi/4)16^2 = 64\,\pi \text{ mm}^2;$$
$$A_c = (\pi/4)\,[50^2 - 40^2] = 225\,\pi \text{ mm}^2$$

Stresses due to tightening of the nuts

As $\Delta = \dfrac{\sigma_c L}{E_c}$ $\therefore$ $0.6 = \dfrac{\sigma_c \times 3000}{100\,000}$ or $\sigma_c = 20$ MPa (compressive)

and as the force in the rod and the tube is the same, $\sigma_s . A_s = \sigma_c . A_c$

or $\sigma_s \times 64\pi = 20 \times 225\,\pi$ or $\sigma_s = 70.3$ MPa (tensile)

Stresses due to temperature rise

As the coefficient of expansion of copper is more than that of steel, it expands more. Thus compressive stress is induced in the copper tube and tensile in the steel rod.

As $\sigma_s . A_s = \sigma_c . A_c$

$\therefore$ $\sigma_s = (A_c/A_s)\,\sigma_c = (225/64)\,\sigma_c = 3.516\,\sigma_c$

Now, from compatibility equation,

Temperature strain of steel + tensile strain of steel

= Temperature strain of copper – compressive strain of copper

i.e. $\alpha_s . L_s t + \dfrac{\sigma_s L_s}{E_s} = \alpha_c L_c t - \dfrac{\sigma_c L_c}{E_c}$.

$$12 \times 10^{-6} \times (3000 + 60) \times 56 + \frac{3.516\,\sigma_c \times 3060}{210\,000}$$
$$= 17 \times 10^{-6} \times 3000 \times 56 - \frac{\sigma_c \times 3000}{100\,000}$$

or $2.056 + 0.051\sigma_c = 2.856 - 0.03\,\sigma_c$

or $0.081\,\sigma_c = 0.8$ or $\sigma_c = 9.87$ MPa

and $\sigma_s = 3.516 \times \sigma_c = 3.516 \times 9.87 = 34.7$ MPa

Final stresses

$\sigma_c = 20 + 9.87 = 29.87$ MPa (compressive)

and $\sigma_s = 70.3 + 34.6 = 104.9$ MPa (tensile)

Example 1.19 *A steel rod of 30-mm diameter is enclosed in a brass tube of 42-mm external diameter and 32-mm internal diameter. Each is 360 mm long and the assembly is rigidly held between two stops 360 mm apart. The temperature of the assembly is then raised by 50°C. Determine*

(i) stresses in the tube and the rod

(ii) stresses in the tube and the rod if the stops yields by 0.15 mm

(iii) yield of the stops if the force at the stops is limited to 60 kN

$E_s = 205$ GPa; $E_b = 90$ GPa; $\alpha_s = 11 \times 10^{-6}$/°C; $\alpha_b = 19 \times 10^{-6}$/°C

Solution Refer Fig. 1.29.

$A_s = (\pi/4)\,30^2 = 225\,\pi \text{ mm}^2$;

$A_b = (\pi/4)\,[42^2 - 32^2]$
$= 185\,\pi\ \text{mm}^2$

(i) When the temperature is raised by 50°,

Stress in the steel rod = $\alpha_s\, t\, E_s$
$= 11 \times 10^{-6} \times 50 \times 205\,000$
$= 112.75$ MPa (compressive)

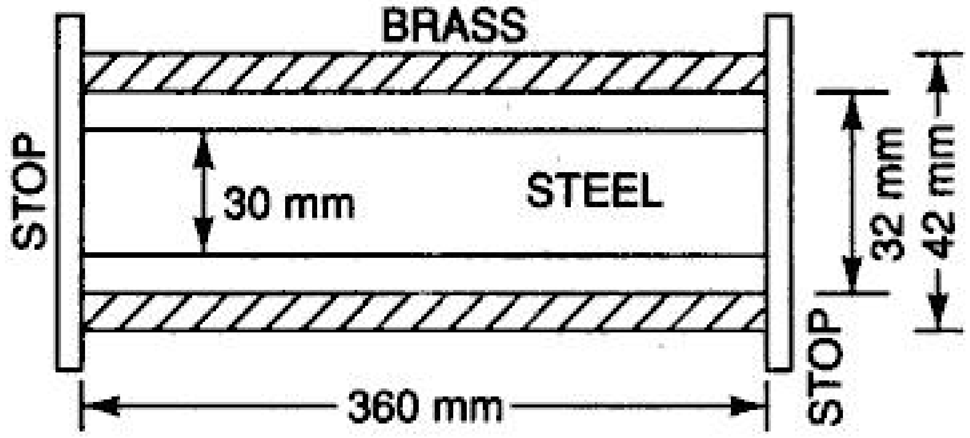


Fig. 1.29

Stress in the brass tube = $\alpha_b\, t\, E_b = 19 \times 10^{-6} \times 50 \times 90\,000$
$= 85.5$ MPa (compressive)

(*ii*) If the stops yields by 0.15 mm, $\Delta_s = (\alpha_s L t - 0.15) = \dfrac{\sigma_s L}{E_s}$

or $$\sigma_s = \alpha_s t E_s - \frac{0.15 E_s}{L} = 112.75 - \frac{0.15 \times 205\,000}{360}$$
$$= 112.75 - 85.42 = 27.33 \text{ MPa (compressive)}$$

and $$\Delta_b = (\alpha_b L t - 0.15) = \frac{\sigma_b L}{E_b}$$

or $$\sigma_b = \alpha_b t E_b - \frac{0.15 E_b}{L} = 85.5 - \frac{0.15 \times 90\,000}{360}$$
$$= 85.5 - 37.5 = 48 \text{ MPa (compressive)}$$

(iii) When the force at the stops is limited to 60 kN, let the yield of the stops be δ,

Then $$\Delta_s = (\alpha_s L t - \delta) = \frac{\sigma_s L}{E_s}$$

or $$\sigma_s = \alpha_s t E_s - \frac{\delta E_s}{L} = 112.75 - \frac{\delta \times 205\,000}{360} = 112.75 - 569.44\,\delta$$

and $$\Delta_b = (\alpha_b L t - \delta) = \frac{\sigma_b L}{E}$$

or $$\sigma_b = \alpha_b t E_b - \frac{\delta E_b}{L} = 85.5 - \frac{\delta \times 90\,000}{360}$$
$$= 85.5 - 250\,\delta$$

Now, Force exerted by steel rod + Force exerted by brass tube = total force on the stops

$$\sigma_s . A_s + \sigma_b . A_b = P$$
$$(112.75 - 569.44\,\delta) \times 225\,\pi + (85.5 - 250\,\delta) \times 185\,\pi = 60\,000$$
$$79\,698 - 402\,513\,\delta + 49\,692 - 145\,299\,\delta = 60\,000$$
$$547\,812\,\delta = 69\,390$$
$$\delta = 0.127 \text{ mm}$$

Example 1.20 *A rigid block AB weighing 180 kN is supported by three rods symmetrically placed as shown in Fig. 1.30. Before attaching the weight, the lower ends of the rods are set at the same level. The areas of cross-section of the steel and copper rods are 800 mm² and 1350 mm² respectively. Determine*

(i) *the stresses in the rods, if the temperature is raised by 25°*
(ii) *the stresses in the rods, if the temperature is raised by 50°*
(iii) *the temperature rise for no stress in the copper rod.*

$E_c = 95$ GPa; $\alpha_c = 18 \times 10^{-6}/°C$;
$E_s = 205$ GPa; $\alpha_s = 11 \times 10^{-6}/°C$

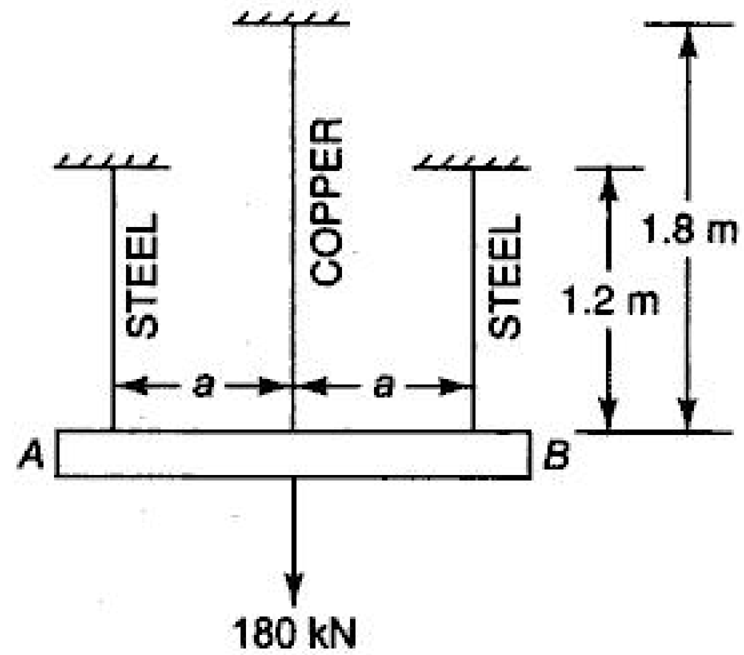


Fig. 1.30

Solution Considering the increase in temperature alone (neglecting the weight of the block), the elongation of copper rod is more as compared to steel rods. On the other hand, if the temperature does not change, there is elongation of all rods and there is tensile stress in all the rods.

Total elongation of each rod is the sum of elongations due to temperature and due to weight. As the block is rigid, it will remain horizontal under all conditions. Thus the total elongation of each rod is the same.

(i) Assume the stress in the copper rod to be compressive, i.e. the force acting upwards.

Then $$\alpha_s L_s t + \frac{P_s L_s}{A_s E_s} = \alpha_c L_c t - \frac{P_c L_c}{A_c E_c}$$

$$11 \times 10^{-6} \times 1200 \times 25 + \frac{P_s \times 1200}{800 \times 205\ 000}$$

$$= 18 \times 10^{-6} \times 1800 \times 25 - \frac{P_c \times 1800}{1350 \times 95\ 000}$$

$$330\ 000 + 7.317 P_s = 810\ 000 - 14.035 P_c$$

$$P_s = 65\ 601 - 1.918 P_c \qquad \text{(i)}$$

From equilibrium equation

$$2P_s - P_c = 180\ 000$$

or $$P_s - 0.5P_c = 90\ 000$$

$$65\ 601 - 1.918 P_c - 0.5 P_c = 90\ 000 \qquad \text{[from (i)]}$$

or $$2.418\ P_c = -24\ 399$$

or $$P_c = -10\ 090 \text{ N (compressive)}$$

and $$P_s = 90\ 000 + 0.5 \times (-10\ 090) = 84\ 955 \text{ (tensile)}$$

$$\sigma_c = -\frac{10\ 090}{1350} = -7.474 \text{ MPa}$$

This shows that the stress in the copper rod is opposite of what was assumed i.e. tensile and not compressive.

$$\sigma_s = \frac{84\ 955}{800} = 106.19 \text{ MPa (tensile)}$$

(ii) If the temperature is raised by 50°,

$$11 \times 10^{-6} \times 1200 \times 50 + 7.317 \times 10^{-6} P_s$$
$$= 18 \times 10^{-6} \times 1800 \times 50 - 14.035 \times 10^{-6} P_c$$
$$660\,000 + 7.317\, P_s = 1\,620\,000 - 14.035 P_c$$
$$P_s = 131\,201 - 1.918\, P_c$$

From equilibrium equation

$$2P_s - P_c = 180\,000 \quad \text{or} \quad P_s - 0.5P_c = 90\,000$$

or $131\,201 - 1.918\, P_c = 90\,000$

or $P_c = 17\,039$ N (compressive)

$$\sigma_c = 17\,039/1350 = 12.62 \text{ MPa (compressive)}$$
$$P_s = 90\,000 + 0.5 \times 17\,039 = 98\,520 \text{ N}$$
$$\sigma_s = 98\,520/800 = 123.1 \text{ MPa (tensile)}$$

(iii) As there is to be no stress and hence no load on the copper rod, $\sigma_c = 0$

Hence load in each rod = 180 000/2 = 90 000 N

$$11 \times 10^{-6} \times 1200 \times t + 7.317 \times 10^{-6} P_s$$
$$= 18 \times 10^{-6} \times 1800 \times t - 0$$
$$13\,200\, t + 7.317\, P_s = 32\,400 t$$
$$19\,200\, t = 7.317 \times 90\,000$$
$$t = 34.3°$$

1.13 SHRINKING ON

A thin tyre of steel or of any other metal can be shrunk on to wheels of slightly larger diameter by heating the tyre to a certain degree which increases its diameter. When the tyre has been mounted and the temperature falls to the normal temperature, the steel tyre tends to come to its original diameter and thus tensile (hoop) stress is set up in the tangential direction.

As shown in Fig. 1.31, let d and D be the diameters of the steel tyre and of the wheel on which the steel tyre is to be mounted (Fig. 1.31), then

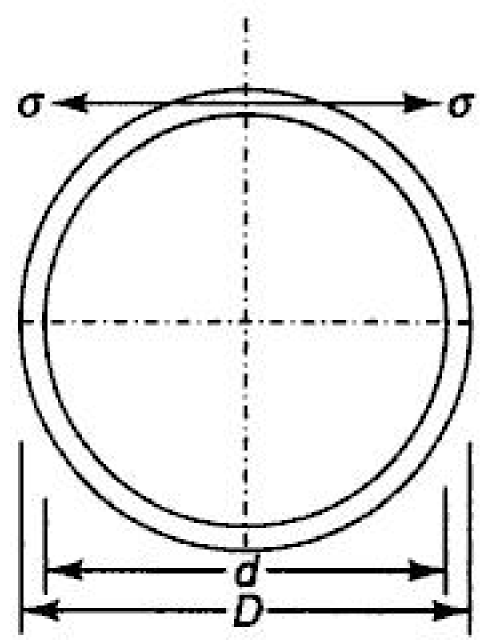


Fig. 1.31

The strain, $$\varepsilon = \frac{\pi D - \pi d}{\pi d} = \frac{D - d}{d}$$

Circumferential tensile stress or hoop stress $= \varepsilon.E = \left(\dfrac{D-d}{d}\right)E$ (1.18)

Example 1.21 *A thin tyre of steel is to be mounted on to a rigid wheel of 1.2-m diameter. Determine the internal diameter of the tyre if the hoop stress is limited to 120 MPa.*

Also determine the least temperature to which the tyre should be heated so that it can be slipped on to the wheel.

$$E_s = 210 \text{ GPa and } \alpha_s = 11 \times 10^{-6}/^\circ C$$

Solution

$$\text{Tensile strain, } \varepsilon = \frac{\pi D - \pi d}{\pi d} = \frac{D-d}{d} = \left(\frac{\sigma}{E}\right)$$

or
$$\frac{D}{d} = \left(\frac{\sigma}{E}\right) + 1 = \frac{\sigma + E}{E} \quad \text{or} \quad \frac{d}{D} = \frac{E}{\sigma + E}$$

or
$$d = \frac{DE}{\sigma + E} = \frac{1200 \times 210\,000}{120 + 210\,000} = 1199.31 \text{ mm or } 1.19931 \text{ m}$$

Increase in the circumferential length = $\pi(D-d)$

Thus $\alpha L t = \pi(D-d)$

or $11 \times 10^{-6} \times (\pi \times 1199.31) \times t = \pi(1200 - 1199.31)$

$t = 52.3^\circ$ C

1.14 STRAIN ANALYSIS

So far, the effect of an axial force on the length of a bar or rod has been considered. In case of a tensile force, the length increases, and in a compressive force, it decreases. However, this axial increase or decrease takes place at the cost of a change in the lateral dimensions of the bar or rod. If an axial tensile force is applied to a bar, its length is increased and its lateral dimensions i.e. the width and breadth or the diameter are decreased (Fig. 1.32). Therefore, any direct stress produces a strain in its own direction as well as an opposite kind of strain in all directions at right angles to its own direction.

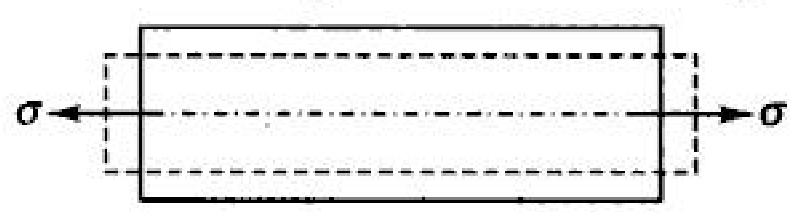


Fig. 1.32

Poisson's Ratio

The ratio of the lateral strain to the longitudinal strain of a material, when it is subjected to a longitudinal stress, is known as *Poisson's ratio* and is denoted by ν. It is found that for elastic materials, the lateral strain is proportional to the longitudinal strain i.e. the ratio of the iateral strain to the longitudinal strain is constant. Thus

$$\frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \text{constant} = \nu \qquad (1.19)$$

The value of ν lies between 0.25 and 0.34 for most of the metals.

Lateral strain = $-\nu \times$ Longitudinal strain = $-\nu . \sigma / E$

(negative sign indicates that it is opposite to the longitudinal strain)

Two-Dimensional Stress System

Consider a system with two pure normal stresses σ_1 and σ_2 as shown in Fig. 1.33.

Strain due to σ_1 in its own direction

$= \sigma_1/E$

Strain due to σ_2 in the direction of

$\sigma_1 = -\nu\sigma_2/E$

Thus, net strain in the direction of σ_1

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E \quad (1.20)$$

In a similar way,

Net strain in the direction of σ_2

$$\varepsilon_2 = \sigma_2/E - \nu\sigma_1/E \quad (1.21)$$

Remember that a tensile stress is taken positive whereas a compressive stress negative.

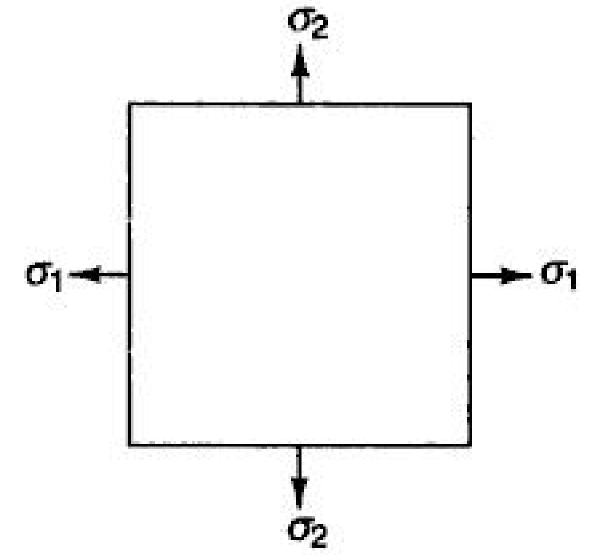


Fig. 1.33

Three-Dimensional Stress System

Let there be a system with three pure normal stresses σ_1, σ_2 and σ_3 as shown in Fig. 1.34.

Strain due to σ_1 in its own direction $= \sigma_1/E$

Strain due to σ_2 in the direction of $\sigma_1 = -\nu\sigma_2/E$

Strain due to σ_3 in the direction of $\sigma_1 = -\nu\sigma_3/E$

Thus, the net strain in the direction of σ_1,

$$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E$$

In a similar way, $\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E$

and $\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E$

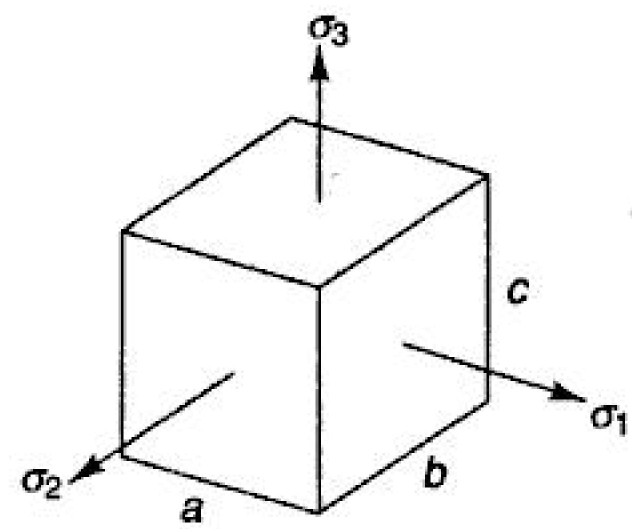


Fig. 1.34

Volumetric Strain

Volumetric strain is defined as the ratio of increase in volume of a body to its original volume when it is acted upon by three mutually perpendicular stresses σ_1, σ_2, and σ_3. For a rectangular solid body of sides a, b and c (Fig. 1.34), let ε_1, ε_2 and ε_3 be the corresponding strains.

Initial volume $= a.b.c$

Final volume $= (a + a\varepsilon_1)(b + b\varepsilon_2)(c + c\varepsilon_3) = abc\,(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{abc\,(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - abc}{abc}$$

$$= (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - 1$$

$$= 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3 - 1$$

$$\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad (1.22)$$

Thus if the products of very small quantities are neglected, the volumetric strain is the algebraic sum of the three mutually perpendicular strains.

In terms of stresses the volumetric strain can be expressed by substituting the values of ε_1, ε_2 and ε_3 from above.

(iii) Let σ be the longitudinal stress to have the same strain,

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.042\ 39}{100} = \frac{\sigma}{205\ 000},\ \sigma = 86.9 \text{ MPa}$$

Longitudinal load = $\sigma \times A = 86.9 \times 35 \times 35 = 106\ 452$ N or 106.452 kN

Example 1.25 *A square steel bar of dimensions 50 mm × 50 mm × 150 mm is subjected to an axial load of 250 kN. Determine the decrease in length of the bar if*

(i) the lateral strain is fully prevented by applying external uniform pressure on the rectangular surfaces.

(ii) only one-third of the lateral strain is prevented by the external pressure.

Solution

(i) $\sigma_1 = 250\ 000/(50 \times 50) = 100$ MPa

Let the compressive stresses applied on the similar lateral sides be σ_2 (= σ_3) to prevent the lateral strain (Fig. 1.35). Then

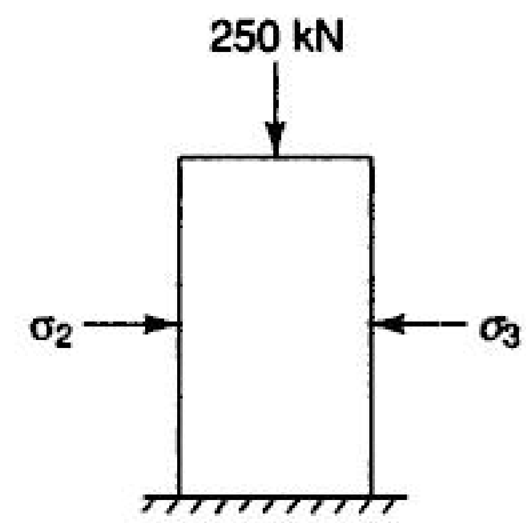


Fig. 1.35

$$\frac{1}{E}(\sigma_2 - v\sigma_3 - v\sigma_1) = 0$$

or $(\sigma_2 - 0.3 \times \sigma_2 - 0.3 \times 100) = 0$ ....$(\sigma_2 = \sigma_3)$

or $0.7\ \sigma_2 = 30$

$\sigma_2 = 42.857$ MPa

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - v\sigma_2 - v\sigma_3) = 0$$

$$= \frac{150}{205\ 000}(100 - 0.3 \times 2 \times 42.857) = 0 \qquad ....(\sigma_2 = \sigma_3)$$

$= 0.054\ 36$ mm

(ii) In the absence of compressive stresses on the sides to prevent the lateral strain,

The lateral strain = $v\sigma_1/E$ (tensile)

Now, one-third of this is to be prevented i.e. $v\sigma_1/3E$ and leaving $2v\sigma_1/3E$ as such. Let the compressive stresses applied on the sides be σ_2.
Then

$$\frac{1}{E}(\sigma_2 - v\sigma_3 - v\sigma_1) = -\frac{2v\sigma_1}{3E}$$

The two strains are of opposite directions.

or $(\sigma_2 - 0.3 \times \sigma_2 - 0.3 \times 100) = -2 \times 0.3 \times 100/3$ $(\sigma_2 = \sigma_3)$

or $0.7\ \sigma_2 = 30 - 20$

$\sigma_2 = 14.286$ MPa

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - v\sigma_2 - v\sigma_3) = 0$$

$$= \frac{150}{205\ 000}(100 - 0.3 \times 2 \times 14.286) = 0 \qquad (\sigma_2 = \sigma_3)$$

$= 0.0669$ mm

From (i) and (ii),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

or $\quad E = 2G(1 + \nu)$

As $\quad E = 3K(1 - 2\nu)$ ...(Eq. 1.27)

$\therefore \quad E = 2G(1 + \nu) = 3K(1 - 2\nu)$ (1.28)

This equation relates the elastic constants.

Also from above, $1 + \nu = \dfrac{E}{2G}, \quad \therefore \quad 2 + 2\nu = \dfrac{E}{G}$ (i)

and $\quad 1 - 2\nu = \dfrac{E}{3K}$ (ii)

Adding (i) and (ii), $3 = E\left(\dfrac{1}{G} + \dfrac{1}{3K}\right) = \dfrac{E}{3KG}(3K + G)$

or $\quad E = \dfrac{9KG}{3K + G}$ (1.29)

Example 1.26 *A bar, 24 mm in diameter and 400 mm in length, is acted upon by an axial load of 38 kN. The elongation of the bar and the change in diameter are measured as 0.165 mm and 0.0031 mm respectively. Determine*

(i) Poisson's ratio, and (ii) the values of the three moduli

Solution

$$A = (\pi/4)\,24^2 = 144\,\pi \text{ mm}^2$$

$$\sigma = 38\,000/144\,\pi = 84 \text{ MPa}$$

Lateral strain = ν .Linear strain

$$\frac{\delta d}{d} = \nu\frac{\delta L}{L} \text{ or } \frac{0.0031}{24} = \nu\frac{0.165}{400} \text{ or } \nu = 0.313$$

$$E = \frac{\sigma}{\varepsilon} = \frac{84}{0.165/400} = 203\,636 \text{ MPa}$$

Also, $\quad E = 2\,G\,(1 + \nu) = 3K\,(1 - 2\nu)$

$$\therefore \quad G = \frac{E}{2(1+\nu)} = \frac{203\,636}{2(1+0.313)} = 77\,546 \text{ MPa}$$

and

$$K = \frac{E}{3(1-2\nu)} = \frac{203\,636}{3(1-2\times 0.313)} = 181\,494 \text{ MPa}$$

Example 1.27 *A bar, 12 mm in diameter, is acted upon by an axial load of 20 kN. The change in diameter is measured as 0.003 mm. Determine*

(i) Poisson's ratio and (ii) the modulus of elasticity and the bulk modulus.

The value of the modulus of rigidity is 80 GPa.

Solution

$$A = (\pi/4)\,12^2 = 36\,\pi \text{ mm}^2$$

$$\sigma = 20\,000/36\,\pi = 176.84 \text{ MPa}$$

24. In the framed structure of Fig. 1.43, the outer rods are of steel and of 260-mm^2 area of cross section whereas the central rod is of brass and of 420-mm^2 area of cross-section. The length of the central rod is 1200 mm. Initially, all the rods are of required length. However, while assembling, the central rod is heated through 40°C. Determine the stresses developed in the rods. E for steel = 205 GPa, E for brass = 85 GPa and α for brass = 19×10^{-6}/°C.

(17.9 kN in 2; 10.34 kN in 1 and 3)

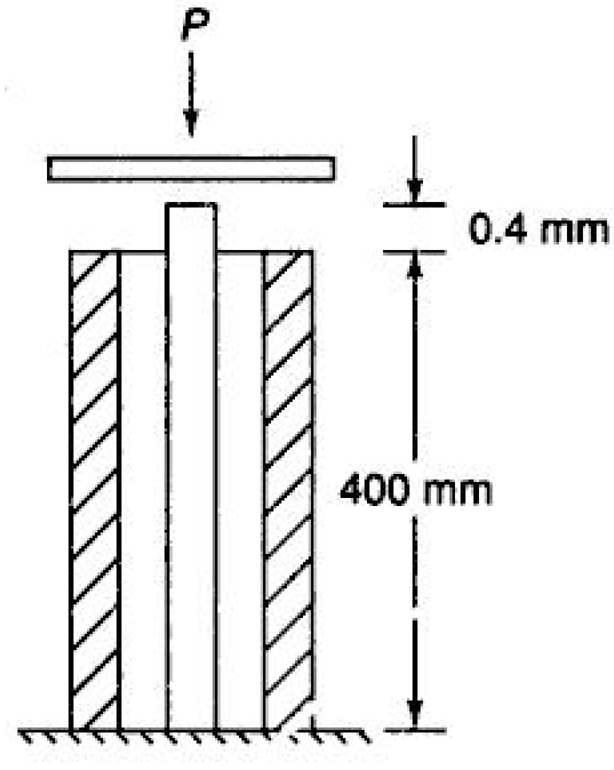


Fig. 1.44

25. A steel sleeve of 24-mm internal diameter and 36-mm external diameter encloses an aluminum rod of 22-mm diameter. The length of the rod is 0.4 mm longer than that of the sleeve which is 400 mm long as shown in Fig. 1.44. Determine
   (i) the compressive load up to which only the rod is stressed
   (ii) the maximum load on the assembly, if the permissible stresses in aluminum and steel are 130 MPa and 175 MPa respectively
   (iii) the deformation of the assembly under maximum load
   E_a = 75 GPa and E_s = 205 GPa

(17.42 kN; 138.71 kN; 0.3416 mm)

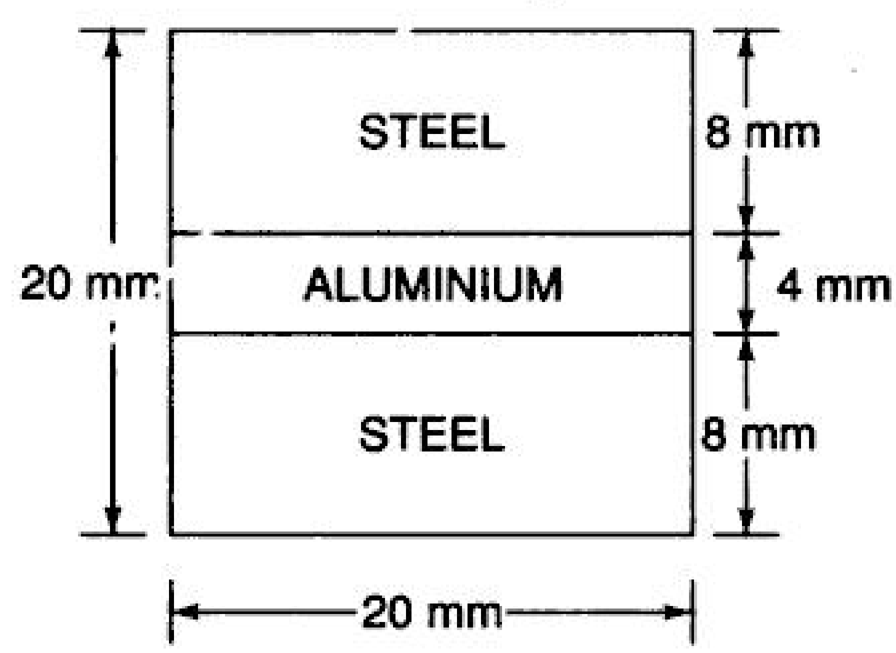


Fig. 1.45

26. A composite bar of 20 mm × 20 mm cross section is made up of three flat bars as shown in Fig. 1.45. All the three bars are rigidly connected at the ends when the temperature is 20°C. Determine
   (i) the stresses developed in each bar when the temperature of the composite bar is raised to 60°C
   (ii) the final stresses in each bar when a load of 17.6 kN is applied to the composite bar

E_a = 80 GPa, $\alpha_a = 11 \times 10^{-6}$/°C

E_s = 200 GPa, $\alpha_s = 22 \times 10^{-6}$/°C

(σ_s = 8 MPa; σ_a = 32 MPa; σ_s = 42 MPa; σ_a = 52 MPa)

27. A load of 120 kN is applied to a bar of 20-mm diameter. The bar which is 400-mm long is elongated by 0.7 mm. Determine the modulus of elasticity of the bar material. If Poisson's ratio is 0.3, find the change in diameter.

(218 GPa; 0.0105 mm)

28. A metallic prismatic specimen is subjected to an axial stress of σ_x and on one pair of sides no constraint is exerted whereas on the other, the lateral strain is

- When $\theta = 90°$, $\sigma_\theta = 0$ and $\tau_\theta = 0$
- When $\theta = 135°$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = \sigma_x/2$ (maximum, clockwise)

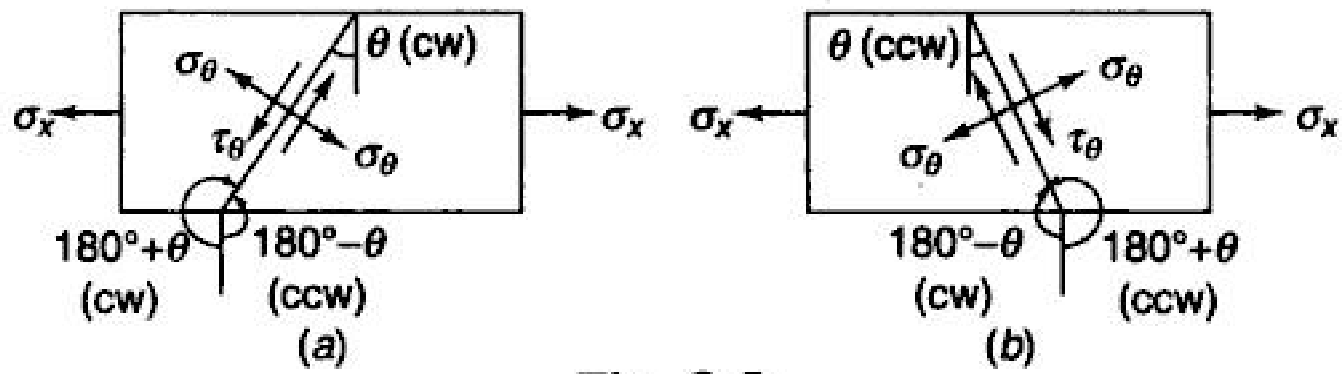


Fig. 2.2

Figures 2.2 (*a*) and (*b*) show the planes inclined at different angles to the vertical alongwith the stresses acting on them. It can be noted from these figures alongwith the above observations that

- a plane at angle θ with the vertical also is the plane with angle $(180° + \theta)$. Thus a plane at angle 45° clockwise with vertical can also be mentioned as the plane at 225° clockwise or 135° counter-clockwise. Similarly, a plane at angle –45° with the vertical would also mean a plane at angle 45° counter-clockwise or angle 225° counter-clockwise or angle 135° clockwise.
- the normal stress on the inclined plane decreases with the increase in angle θ, from maximum on the vertical plane to zero on the horizontal plane.
- the shear stress is negative (counter-clockwise) between 0° and 90° and positive (clockwise) between 0° and – 90°. Remember that plane at 135° to the vertical also means a plane at – 45° as described above.
- the maximum shear stress is equal to one half the applied stress.

The resultant stress on the plane *AC*,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sigma_x\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}$$

$$= \sigma_x \cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$

$$= \sigma_x \cos\theta \tag{2.3}$$

Inclination with the normal stress,

$$\tan\varphi = \frac{\sigma_x \sin\theta\cos\theta}{\sigma_x\cos^2\theta} = \tan\theta$$

or

$$\varphi = \theta \tag{2.4}$$

That is, it is always in the direction of the applied stress.

(ii) Bi-axial Stress Condition

Let an element of a body be acted upon by two tensile stresses acting on two perpendicular planes of the body as shown in Fig. 2.3. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively.

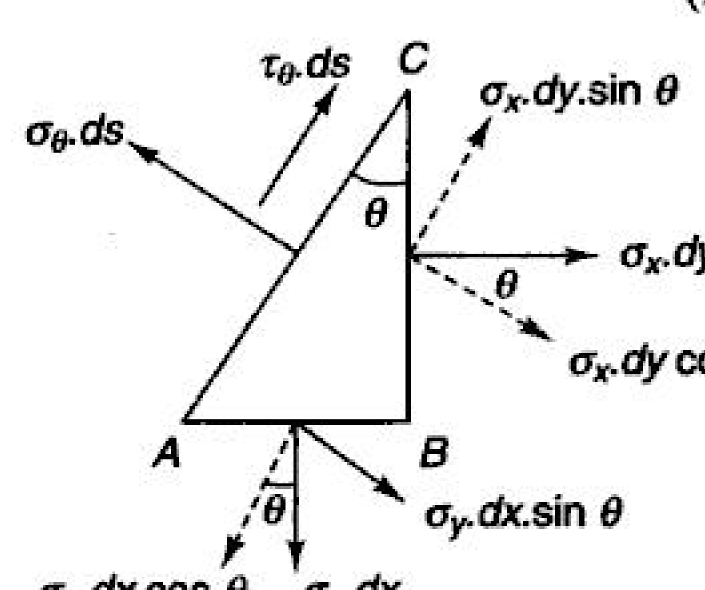


Fig. 2.3

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

The above equations show that

- the normal stress is positive (tensile) when θ is between 0° and 90° and negative (compressive) between 90° and 180°. Maximum values being at 45° (= τ) and 135° (= $-\tau$)
- the shear stress is positive (clockwise) for $\theta < 45°$ and negative (counter- clockwise) for $\theta > 45°$ and < 135° and again positive between $\theta > 135°$ and < 180°.
- the shear stress is zero at 45° and 135° where the normal stress is maximum.

These conclusions indicate that when a body is acted upon by pure shear stresses on two perpendicular planes, the planes inclined at 45° are subjected to a tensile stress of magnitude equal to that of the shear stress while the planes inclined at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes.

Compare this result with Eq. 2.12.

(iv) Bi-axial and Shear Stresses Condition

Let an element of a body be acted upon by two tensile stresses alongwith shear stresses acting on two perpendicular planes of the body as shown in Fig. 2.5. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively.

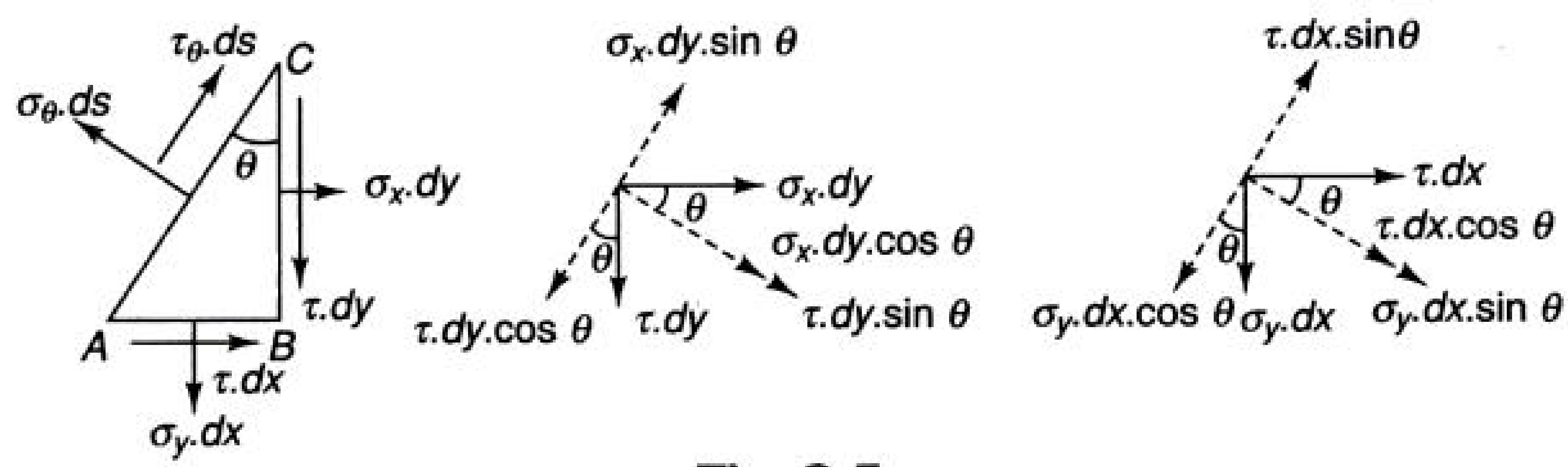


Fig. 2.5

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta ds - \sigma_x .dy.\cos\theta - \sigma_y .dx.\sin\theta - \tau.dy.\sin\theta - \tau.dx.\cos\theta$$

or

$$\sigma_\theta = \frac{\sigma_x dy\cos\theta}{ds} + \frac{\sigma_y dx\sin\theta}{ds} + \frac{\tau.dy\sin\theta}{ds} + \frac{\tau.dx\cos\theta}{ds}$$

$$= \frac{\sigma_x dy\cos\theta}{dy/\cos\theta} + \frac{\sigma_y dx\sin\theta}{dx/\sin\theta} + \frac{\tau dy\sin\theta}{dy/\cos\theta} + \frac{\tau\, dx\cos\theta}{dx/\sin\theta}$$

$$= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta$$

$$= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + \tau\sin 2\theta \quad (2.19)$$

$$= \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) + \tau.\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \quad (2.20)$$

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Maximum (Principal) Shear Stress

In any complex system of loading, the maximum and the minimum normal stresses are the principal stresses and the shear stress is zero in their planes. To find the maximum value of shear stress and its plane in such a system, consider the equation for shear stress in a plane, i.e.

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta \qquad \text{(Eq. 2.21)}$$

For maximum value of τ_θ, differentiate it with respect to θ and equate to zero,

$$\frac{d\tau_\theta}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau\sin 2\theta = 0$$

or $$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau} \qquad (2.33)$$

This indicates that there are two values of 2θ differing by 180° or two values θ differing by 90°. Thus maximum shear stress planes lie at right angles to each other.

Now, as $\tan 2\theta = -\frac{(\sigma_x - \sigma_y)}{2\tau}$ can be represented as shown in Fig. 2.7

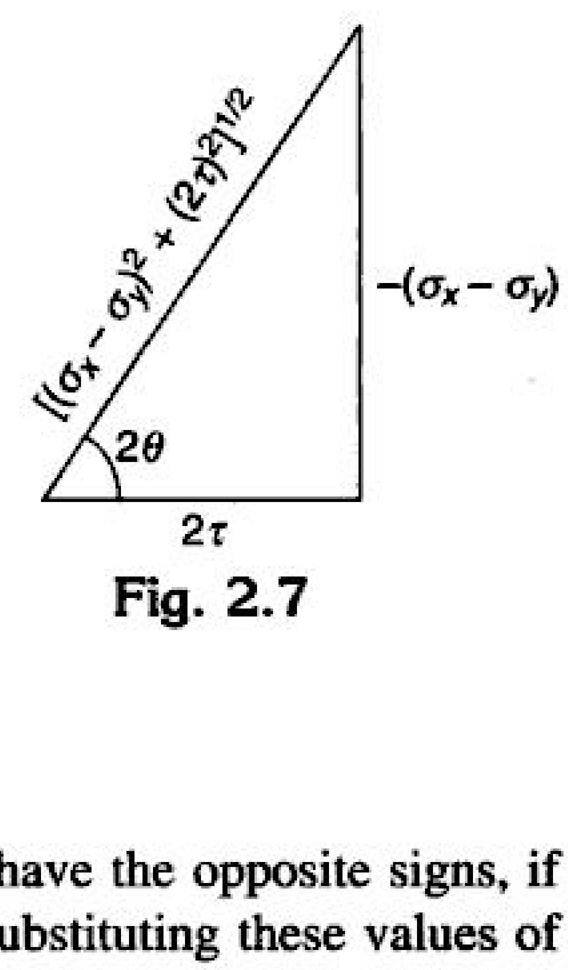

Fig. 2.7

$$\sin 2\theta = \mp\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}};$$

$$\cos 2\theta = \pm\frac{2\tau}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$

Right-hand sides of both the above equations should have the opposite signs, if one is positive the other is negative while using them. Substituting these values of sin 2θ and cos 2θ in Eq. 2.21, two values of the shear stress are obtained.

$$\therefore \quad \tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta$$

$$= -\left(\mp\frac{1}{2}(\sigma_x - \sigma_y)\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}\right) \pm \tau.\frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} = \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

This provides maximum and minimum values of shear stress, both numerically equal. In fact the negative or minimum value indicate that it is at right angle to the positive value as discussed above and two are the complimentary shear stresses. Thus magnitude of the maximum or principal shear stress is given by

- Bisect LM at C and draw a circle with C as centre and radius equal to CR (= CS). Let $\angle LCR = \beta$.
- Rotate the radial line CR through angle 2θ in the clockwise direction if θ is taken clockwise and let it take the position CP.
- Draw $NP \perp$ on the x-axis. Join OP.

It can be proved that ON and NP represent the normal and the shear stress components on the inclined plane AD.

From the geometry of the figure,

$$OC = \frac{1}{2}(\sigma_x + \sigma_y) \text{ as before.}$$

$$\begin{aligned} CN &= CP \cos(2\theta - \beta) \\ &= CR \cos(2\theta - \beta) \qquad \ldots(CP = CR) \\ &= CR(\cos 2\theta \cos\beta + \sin 2\theta \sin\beta) \\ &= (CR\cos\beta)\cos 2\theta + (CR\sin\beta)\sin 2\theta \\ &= CL\cos 2\theta + LR\sin 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta \qquad \ldots(CL = OL - OM) \end{aligned}$$

Thus $ON = OC + CN = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau.\sin 2\theta = \sigma_\theta$

...(Eq. 2.20)

and

$$\begin{aligned} NP &= CP\sin(2\theta - \beta) = CR\sin(2\theta - \beta) \\ &= CR(\sin 2\theta\cos\beta - \cos 2\theta\sin\beta) \\ &= (CR\cos\beta)\sin 2\theta - (CR\sin\beta)\cos 2\theta \\ &= CL\sin 2\theta - LR\cos 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta = -\tau_\theta \qquad \ldots\text{(Eq. 2.21)} \end{aligned}$$

As NP is below the x-axis, therefore, the shear stress is negative or counter-clockwise.

Mathematically,
$$\begin{aligned} NP &= -\left[\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau\cos 2\theta\right] \\ &= -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta] \end{aligned}$$

Principal Stresses

As shear stress is zero on principal planes, OF represents the major principal plane with maximum normal stress. In a similar way, OE represents the minor principal plane.

$$\begin{aligned} OF &= OC + CF = OC + CR = OC + \sqrt{CL^2 + LR^2} \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + \tau^2} \\ &= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \\ &= \text{Major principal stress} \end{aligned}$$

- Make angle $FCR = 60^o$, i.e. double the angle of the inclined plane with *OF* in the clockwise direction.

Then *CR* represents the inclined plane.

$\sigma_{30} = OL = 90$ MPa (tensile)

$\tau_{30} = LR = 17.32$ MPa (counter-clockwise)

$\sigma_r = OR = 91.65$ MPa

Inclination of the resultant with *OL* or σ_{30}, $\varphi = 10.9^o$

The results are shown in Fig. 2.11(*b*).

(b) $$\sigma_{30} = 100\cos^2 30^o - 60\sin^2 30^o = 100\times\frac{3}{4} - 60\times\frac{1}{4} = 60 \text{ MPa (tensile)}$$

$$\tau_{30} = -\frac{1}{2}[100-(-60)]\sin 60^o = -80\times 0.866 = -69.28 \text{ MPa (ccw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{60^2 + (-69.28)^2} = 91.65 \text{ MPa}$$

inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.11^\circ$

α can be found to be -19.11°.

Solution by Mohr's circle is shown in Fig. 2.12 which is self-explanatory.

$\sigma_{30} = OL = 60$ MPa (tensile)

$\tau_{30} = LR = 69.3$ MPa (counter-clockwise)

$\sigma_r = OR = 91.65$ MPa

Inclination of the resultant with *OL* or σ_{30}, $\varphi = 49.1^o$

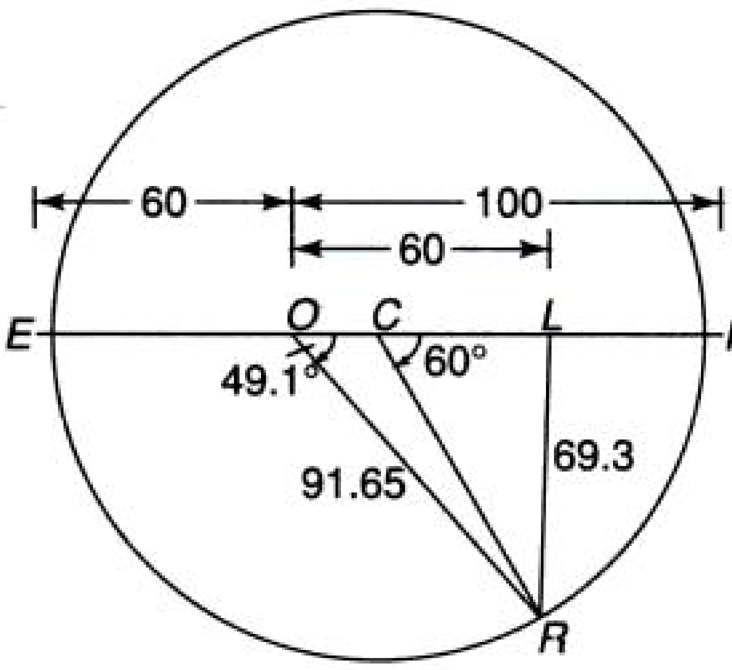

Fig. 2.12

(c) $$\sigma_{30} = -100\cos^2 30^o + 60\sin^2 30^o = -100\times\frac{3}{4} + 60\times\frac{1}{4} = -60 \text{ MPa (comp.)}$$

$$\tau_{30} = -\frac{1}{2}(-100-60)\sin 60^o = 80\times 0.866 = 69.28 \text{ MPa (cw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{(-60)^2 + (69.28)^2} = 91.65 \text{ MPa}$$

Inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.11^o$

α can be found to be -19.11^o.

(d) $$\sigma_{30} = -100\cos^2 30^o - 60\sin^2 30^o = -100\times\frac{3}{4} - 60\times\frac{1}{4} = -90 \text{ MPa (comp.)}$$

Thus in a plane at 60°,

$$20 = \frac{1}{2}(-50+\sigma_y)+\frac{1}{2}(-50-\sigma_y)\cos 120^\circ + 30\sin 120^\circ$$

$$40 = -50+\sigma_y + (-50-\sigma_y)(-0.5) + 60\times 0.866$$

$$40 = -50+\sigma_y + 25 + 0.5\sigma_y + 51.96$$

$$1.5\,\sigma_y = 13.04 \quad \text{or} \quad \sigma_y = 8.69 \text{ MPa}$$

$$\tau_\theta = -(-50-8.69)\sin 120^\circ + 30\cos 120^\circ = 10.414 \text{ MPa}$$

$$\text{Principal stress} = \frac{1}{2}(\sigma_x+\sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau^2}$$

$$= \frac{1}{2}(-50+8.69) \pm \frac{1}{2}\sqrt{(-50-8.69)^2+4(30)^2}$$

$$= -20.66 \pm 41.97 = 21.31 \text{ and } -62.63 \text{ MPa}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x-\sigma_y} = \frac{2\times 30}{-50-8.69} = -1.0223$$

or $\quad 2\theta = -45.63$ or $\theta_1 = -22.82^\circ$ or 157.18°

and $\quad \theta_2 = -22.82^\circ + 90^\circ = 67.18^\circ$

Example 2.7 *Figure 2.15(a) shows the resultant stresses on two planes at a certain point in a material. On a certain plane it is 800 MPa compressive at an angle of 30° to its normal and on another plane it is 600 MPa tensile at an angle of 75° to its normal. Determine the angle between the planes. Also find the principal stresses and their directions to the given plane.*

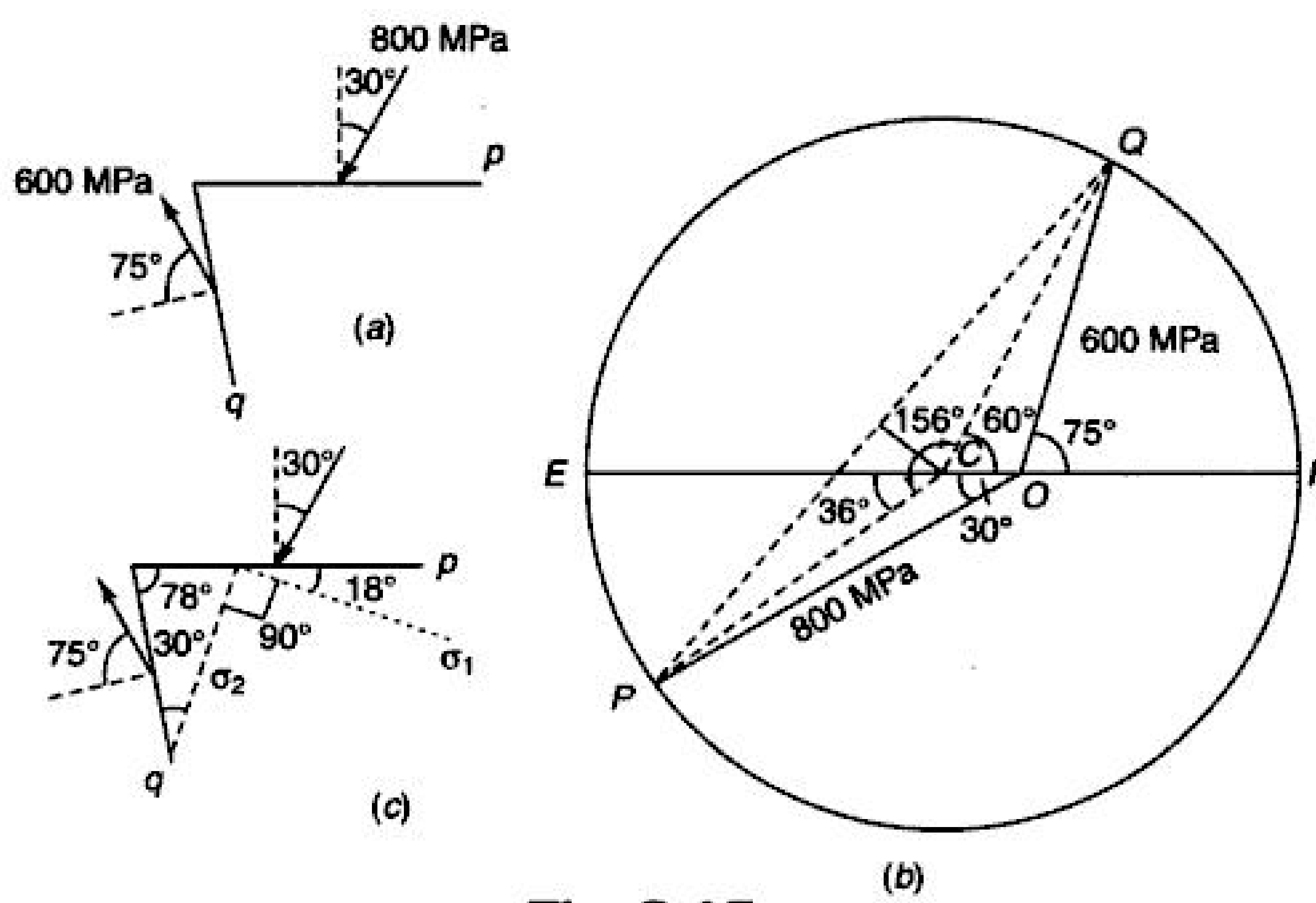

Fig. 2.15

Solution On the plane p, the resultant stress is 800 MPa compressive, its normal component will be compressive and the shear component counter-clockwise.

2.6 ELLIPSE OF STRESS

This is another graphical method to be used when a material is subjected to direct stresses σ_x and σ_y. The method is as follows:

- Draw two circles with O as centre and radii equal to σ_x and σ_y taken to a suitable scale (Fig. 2.19).
- Through O draw AB parallel to the inclined plane.
- Draw $OE \perp AB$ through O intersecting the inner circle at D and outer circle at E.
- Draw $EG \perp OX$.
- Draw $DG \perp EF$.

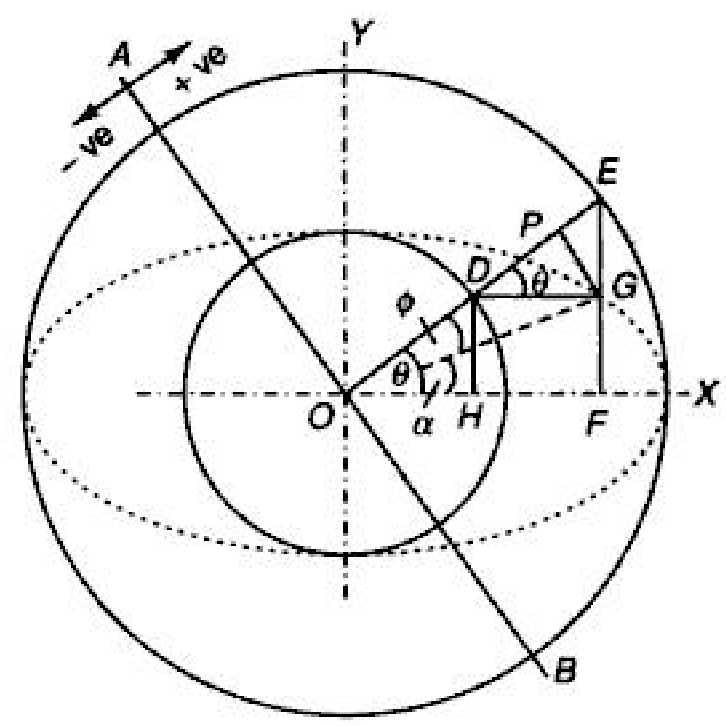

Fig. 2.19

Now, $OP = OD + DP = OD + DG \cos\theta$

$= \sigma_y + (DE\cos\theta)\cos\theta = \sigma_y + (\sigma_x - \sigma_y)\cos^2\theta$

$= \sigma_x \cos^2\theta + \sigma_y(1 - \cos^2\theta) = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta = \sigma_\theta$

(Refer Eq. 2.5)

and $PG = DG\sin\theta = (DE\cos\theta)\sin\theta$

$$= (\sigma_x - \sigma_y)\cos\theta\sin\theta = \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau_\theta$$

(Refer Eq. 2.24 with $\tau = 0$)

Also, $OF = OE\cos\theta = \sigma_x\cos\theta$

and $FG = HD\cos\theta = \sigma_y\sin\theta$

$$\therefore \quad OG = \sqrt{\sigma_x^2\cos^2\theta + \sigma_y^2\sin^2} = \sigma_r$$

(Refer Eq. 2.8)

$$\tan\alpha = \frac{\sigma_y\sin\theta}{\sigma_x\cos\theta} = \frac{\sigma_y}{\sigma_x}\tan\theta$$

(Refer Eq. 2.11)

$$\tan\varphi = \frac{PG}{OP} = \frac{(\sigma_x - \sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta + \sigma_y\sin^2\theta}$$

(Refer Eq. 2.9)

For different values of θ, point G can be located, the locus of which is evidently an ellipse as shown in the figure. The diagram is thus known as ellipse of stress.

Example 2.10 *A piece of material is subjected to two perpendicular stresses as follows:*

(a) Tensile stresses of 100 MPa and 60 MPa

(b) Tensile stress of 100 MPa and compressive stress of 60 MPa

Determine normal and tangential stresses on a plane inclined at 30° to the plane of 100 MPa stress. Also find the resultant and its inclination with the normal stress using ellipse stress method.

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Example 2.11 *Figure 2.24(a) shows the strains in three directions p, q and r in a plane, the magnitudes being 600×10^{-6}, -150×10^{-6} and 250×10^{-6}. Determine the magnitude and direction of the principal strains in this plane.*

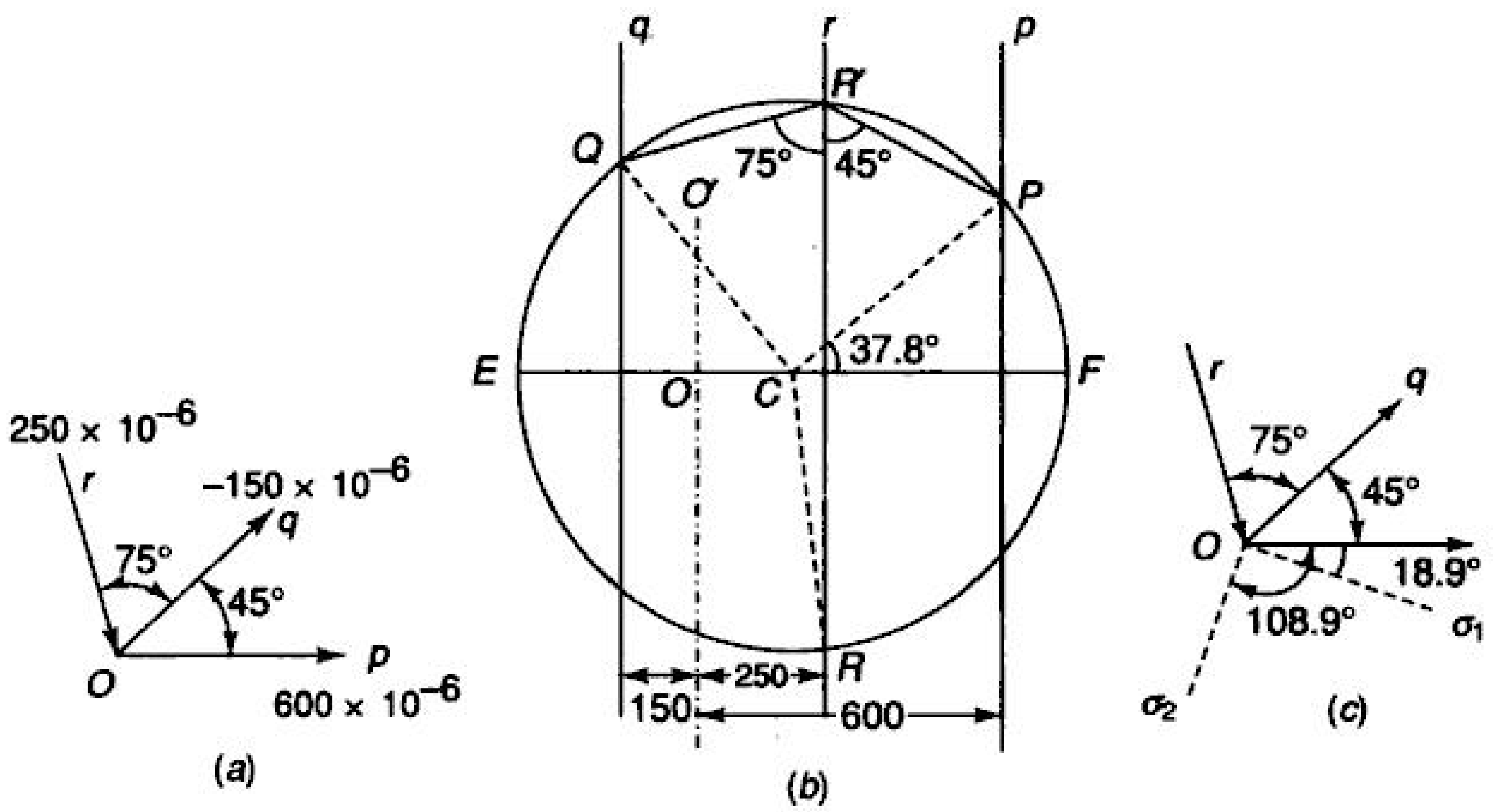

Fig. 2.24

Assuming no stress in a plane perpendicular to this plane, find the principal stresses at the point. Take E = 205 GPa and ν = 0.3.

Solution

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta \quad \text{(Eq. 2.40)}$$

$$\therefore \quad \varepsilon_0 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) = 600 \times 10^{-6}$$

$$\text{or} \quad \varepsilon_x = 600 \times 10^{-6} \quad \text{(i)}$$

$$\varepsilon_{45} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 90^\circ + \frac{1}{2}\varphi \sin 90^\circ$$

$$-150 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}\varphi$$

$$\text{or} \quad \varepsilon_y + \varphi = -900 \times 10^{-6} \quad \text{(ii)}$$

$$\varepsilon_{120} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 240^\circ + \frac{1}{2}\varphi \sin 240^\circ$$

$$250 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) - \frac{1}{4}(600 \times 10^{-6} - \varepsilon_y) - \frac{\sqrt{3}}{4}\varphi$$

$$= \frac{1}{4}600 \times 10^{-6} + \frac{3}{4}\varepsilon_y - \frac{\sqrt{3}}{4}\varphi$$

$$\varepsilon_y - 0.577\,\varphi = 133.3 \times 10^{-6} \quad \text{(iii)}$$

17. A piece of material is acted upon by tensile stresses of 50 MPa and 25 MPa at right angle to each other. Determine by ellipse of stress, the magnitude and direction of the resultant stress on a plane at 45° to the 50 MPa stress.
(39.5 MPa, 18° with normal stress, 27° with 100 MPa stress)

18. The stresses at a point in three coplanar directions are measured as σ_0 = 80 MPa (tensile), σ_{60} = 400 MPa (tensile) and σ_{120} = 200 MPa (compressive) where subscripts indicates the relative angular position of the planes in degrees. Determine the principal stresses and the planes.
[449 MPa (tensile) at 14° to 400 MPa and 251 MPa (compressive) at 16° to 200 MPa stress]

19. The readings of a strain gauge rosette inclined at 45° with each other are 4×10^{-6}, 3×10^{-6} and 1.6×10^{-6}, the first gauge being along x-axis. Determine the principal strains and the planes.
(4.04×10^{-6}, 1.58×10^{-6} ; 5° and 95°)

3.4 SHEAR STRAIN ENERGY

Consider a block with dimensions L, b and h as shown in Fig. 3.5. Assume it to be rigidly fixed to the ground. A shear force P is applied gradually along the top surface.

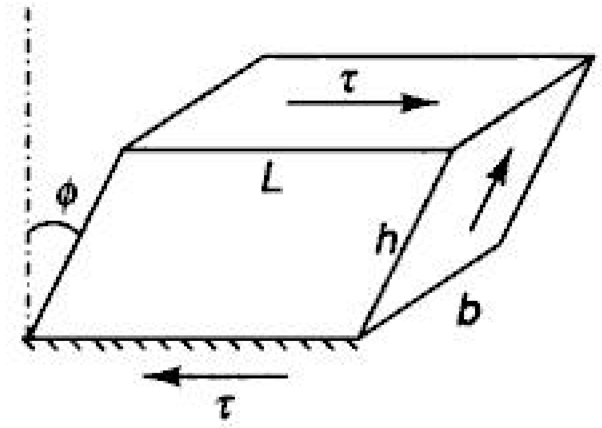

Fig. 3.5

Strain energy, U = Work done in straining = $\frac{1}{2}\times$ Final couple × Angle turned

$$= \frac{1}{2} \times \text{Final force} \times h \times \varphi$$

$$= \frac{1}{2} \times (\text{Shear stress} \times \text{Area}) \times h \times \varphi$$

$$= \frac{1}{2}.\tau.(L.b).h.\frac{\tau}{G} \qquad \text{....... (As } G = \tau/\varphi \text{ or } \varphi = \tau/G\text{)}$$

$$= \frac{\tau^2}{2G}.(L.b.h)$$

$$= \frac{\tau^2}{2G} \times \text{Volume} \qquad (3.9)$$

or Shear strain energy per unit volume = $\frac{\tau^2}{2G}$ (3.10)

It is similar to $\sigma^2/2E$ for direct stress.

3.5 SHEAR STRAIN ENERGY (THREE-DIMENSIONAL STRESS SYSTEM)

Consider a unit cube acted upon by three principal stresses σ_1, σ_2 and σ_3 as before. The total work done by the external forces cause

- change of volume due to application of direct stresses and
- distortion due to shearing stresses which do not affect the volumetric change.

Thus,

Total strain energy = Volumetric strain energy + Shear strain energy

Now total strain energy,

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \text{ per unit volume} \qquad \text{(Eq.3.5)}$$

Volumetric strain energy,

$$U_v = \frac{1}{2}\text{Average stress} \times \text{Volumetric strain}$$

$$= \frac{1}{2}\left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right) \times \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E} \qquad \text{(Using Eq. 1.23)}$$

$$= \frac{1}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2(1 - 2\nu) \qquad (3.11)$$

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Similarly,

(iii) $\frac{U}{U_1} = \frac{0.6667/2 + 0.3333}{2 \times 0.6667 + 0.3333} = 0.4$ (iv) $\frac{U}{U_1} = \frac{0.5/2 + 0.5}{2 \times 0.5 + 0.5} = 0.5$

(v) $\frac{U}{U_1} = \frac{0/2 + 1}{2 \times 0 + 1} = 1$

Example 3.4 *The cross-sections of two bars A and B made up of the same material and each 320-mm long are as follows:*

- *Bar A: 24-mm diameter for a length of 80 mm and 48 mm for the remaining 240 mm*
- *Bar B: 24-mm diameter for a length of 240 mm and 48 mm for the remaining 80 mm*

An axial blow to bar A produces a maximum instantaneous stress of 160 MPa. Determine the

(i) maximum instantaneous stress produced by the same blow to bar B.

(ii) ratio of energies stored by the two bars when subjected to maximum permissible stress.

(iii) ratio of energies per unit volume of the two bars when subjected to maximum permissible stress.

Solution (Refer Fig. 3.7.)

For the same blow to bar *B*, the strain energy produced by the blow should equal to that produced by the blow to the first bar.

In bar A,

Maximum instantaneous stress in the smaller cross-section = 160 MPa

Maximum instantaneous stress in the larger cross-section = $160 \times \left(\frac{24}{48}\right)^2 = 40$ MPa

In bar B,

Let maximum instantaneous stress in the smaller cross-section = σ

Then maximum instantaneous stress in the larger cross-section = $\sigma/4$

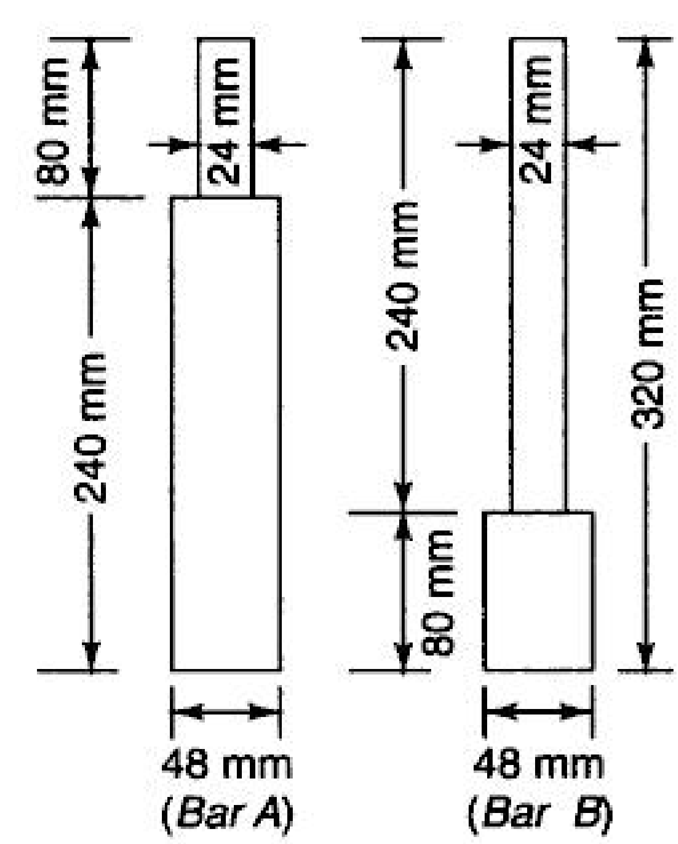

Fig. 3.7

Strain energy of bar *A* = strain energy of bar *B*

$$\frac{40^2}{2E} \times \frac{\pi}{4}(48)^2 \times 240 + \frac{160^2}{2E} \times \frac{\pi}{4}(24)^2 \times 80$$
$$= \frac{(\sigma/4)^2}{2E} \times \frac{\pi}{4}(48)^2 \times 80 + \frac{\sigma^2}{2E} \times \frac{\pi}{4}(24)^2 \times 240$$

Dividing throughout by $\frac{\pi}{4} \times 24^2 \times \frac{80}{2E}$,

Example 3.7 *A lift is operated by three ropes each having 28 wires of 1.4 mm diameter. The cage weighs 1.2 kN and the weight of the rope is 4.2 N/m length. Determine the maximum load carried by the lift if each wire is of 36 m length and the lift operates (i) without any drop (ii) with a drop of 96 mm during operations.*

E (rope) = 72 GPa and allowable stress = 115 MPa

Solution Total area of cross section, $A = \frac{\pi}{4}.(1.4)^2 \times 3 \times 28 = 129.3 \text{ mm}^2$

The maximum stress occurs at the top of the wire rope where the weight of the rope is maximum.

Thus maximum load = weight of cage + weight of rope

$= 1200 + 3 \times 36 \times 4.2 = 1653.6$ N

Initial stress in the rope, $\sigma = \frac{1653.6}{129.3} = 12.8$ MPa

Equivalent static stress available for carrying the load = 115 − 12.8 = 102.2 MPa

Thus, equivalent static load that can be carried,

$P_e = 102.2 \times 129.3 = 13\,214$ N

The extension of the rope, $\Delta = \frac{102.2 \times 36\,000}{72\,000} = 51.1$ mm

(i) With no drop, Let W be the weight which can be applied suddenly, $W.\Delta = \frac{1}{2} P.\Delta$

or $W = 13214/2 = 6607$ N or 6.607 kN

(ii) With 96 mm drop, Let W be the weight,

$$W(h + \Delta) = \frac{1}{2} P.\Delta \quad \text{or} \quad W(96 + 51.1) = \frac{1}{2} \times 13214 \times 51.1$$

or $W = 2295$ N or 2.295 kN

Example 3.8 *A vertical composite tie bar rigidly fixed at the upper end consists of a steel rod of 16-mm diameter enclosed in a brass tube of 16-mm internal diameter and 24-mm external diameter, each being 2 m long. Both are fixed together at the ends. The tie bar is suddenly loaded by a weight of 8 kN falling through a distance of 4 mm. Determine the maximum stresses in the steel rod and the brass tube.*

E_s = 205 GPa and E_b = 100 GPa

Solution Refer Fig. 3.9.

$A_s = (\pi/4)16^2 = 64\,\pi$, $\quad A_b = (\pi/4)(24^2 - 16^2) = 80\,\pi$

Let x = Extension of bar in mm

$$\sigma_s = \frac{E_s.x}{L} \text{ and } \sigma_b = \frac{E_b.x}{L}$$

Strain energy of the bar $= \frac{\sigma_s^2}{2E_s} A_s L + \frac{\sigma_b^2}{2E_b} A_b L$

$$= \frac{E_s^2 x^2}{L^2.2E_s} A_s L + \frac{E_b^2 x^2}{L^2.2E_b} A_b L$$

$= 100^2 + 50^2 + (-25)^2 - 2 \times 0.3 \times [100 \times 50 + 50 \times (-25) + (-25) \times 100]$

$= 13\ 125$

$\sigma = 114.6$ MPa

Factor of safety = 220/114.6 = 1.92

(v) *Maximum shear strain energy theory*

$$2\sigma^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$

$$= (100 - 50)^2 + (50 + 25)^2 + (-25 - 100)^2 = 23\ 750$$

$\sigma^2 = 11\ 875$ $\qquad \sigma = 108.97$

Factor of safety = 220/108.97= 2.02

Example 3.10 *A bolt is acted upon by an axial pull of 16 kN alongwith a transverse shear force of 10 kN. Determine the diameter of the bolt required according to different theories. Elastic limit of the bolt material is 250 MPa and a factor of safety 2.5 is to be taken. Poisson's ratio is 0.3.*

Solution The permissible stress in simple tension = 250/2.5 = 100 MPa

Let the required area of cross-section and the diameter of the bolt be a and d respectively under different theories.

The applied tensile stress = 16 000/a

The applied shear stress = 10 000/a

Maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$

$$= \frac{1}{2a}(16\ 000) + \frac{1}{2a}\sqrt{16\ 000^2 + 4 \times 10\ 000^2} \quad \ldots(\sigma_y = 0)$$

$$= (8000 + 12\ 806)/a = 20\ 806/a \text{ (tensile)}$$

Minimum principal stress, $\sigma_2 = (8000 - 12\ 806)/a = 4806/a$ (compressive)

(i) *Maximum principal stress theory:*

Maximum principal stress, $\sigma_1 = 20\ 806/a$

Thus $\quad 20\ 806/a = 100$

$$\frac{\pi}{4}d^2 = 208.06 \text{ or } d = 16.28 \text{ mm}$$

(ii) *Maximum shear stress theory*

Maximum shear stress = $[20\ 806 - (-4806)]/2a = 12\ 806/a$

Maximum shear stress in simple tension = 100/2 = 50 MPa

$\therefore \quad 12\ 806/a = 50$

$$\frac{\pi}{4}d^2 = 256.12 \text{ or } d = 18.05 \text{ mm}$$

(iii) *Maximum principal strain theory*

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = [20\ 806 - 0.3 \times (-4806)]/a = 22\ 247.8/a \quad \ldots(\sigma_3 = 0)$$

$$22\ 247.8/a = 222.48$$

$$\frac{\pi}{4}d^2 = 222.48 \text{ or } d = 16.83 \text{ mm}$$

4. Derive expressions for the strain energy in a three-dimensional stress system.
5. What is shear strain energy? Find its value per unit volume of the material.
6. Derive the relation for shear strain energy for a three-dimensional stress system.
7. What is the value of maximum stress induced in a body when the load is applied suddenly?
8. Deduce the relation for stress in case of impact and shock loading.
9. What are the main theories of failure for a material? Explain their relative use.
10. Give an account of graphical representation of various theories of failure.
11. A load of 22 kN is lowered by a steel rope at the rate of 750 mm/s. The diameter of the rope is 28 mm. When the length of the rope unwound is 12 m, the rope suddenly gets jammed. Find the instantaneous stress developed in the rope. Also calculate the instantaneous elongation of the rope. $E = 205$ GPa. (187.1 MPa, 10.95 mm)
12. A weight of 2 kN falls 24 mm on to a collar fixed to a steel bar that is 14 mm in diameter and 5.5 m long. Determine the maximum stress induced in the bar. $E_s = 205$ GPa. (166 MPa)
13. A weight of 800 N falls 30 mm on to a collar fixed to a steel bar of 1.2-m length. The steel bar is of 24-mm diameter for half of its length and 12 mm for the rest half. Determine the maximum stress and the extension in the bar. $E_s = 205$ GPa. (347.7 MPa; 1.272 mm)
14. A lift is operated by two 20-m long ropes and consisting of 30 wires of 1.5-mm diameter. The weight of the cage is 1 kN and the rope weighs 3.6 N/m length. Determine the maximum load that the lift can carry if it drops through 120 mm during operations. E (rope) = 78 GPa and allowable stress = 125 MPa. (1.188 kN)
15. A vertical tie rod consists of a 3-m long and of 24-mm diameter steel rod encased throughout in a brass tube of 24-mm internal diameter and 36-mm external diameter. The rod is rigidly fixed at the top end. The composite tie rod is suddenly loaded by a weight of 13.5 kN falling freely through 6 mm before being stopped by the tie. Determine the maximum stresses in steel and the brass. $E_s = 205$ GPa and $E_b = 98$ GPa. (143.8 MPa; 68.76 MPa)
16. An axial pull of 20 kN alongwith a shear force of 15 kN is applied to a circular bar of 20 mm diameter. The elastic limit of the bar material is 230 MPa and the Poisson's ratio, $\nu = 0.3$. Determine the factor of safety against failure based on
    (a) maximum shear stress theory
    (b) maximum strain energy theory
    (c) maximum principal strain energy theory
    (d) maximum shear strain energy theory (2; 2.3; 2.37; 2.2)

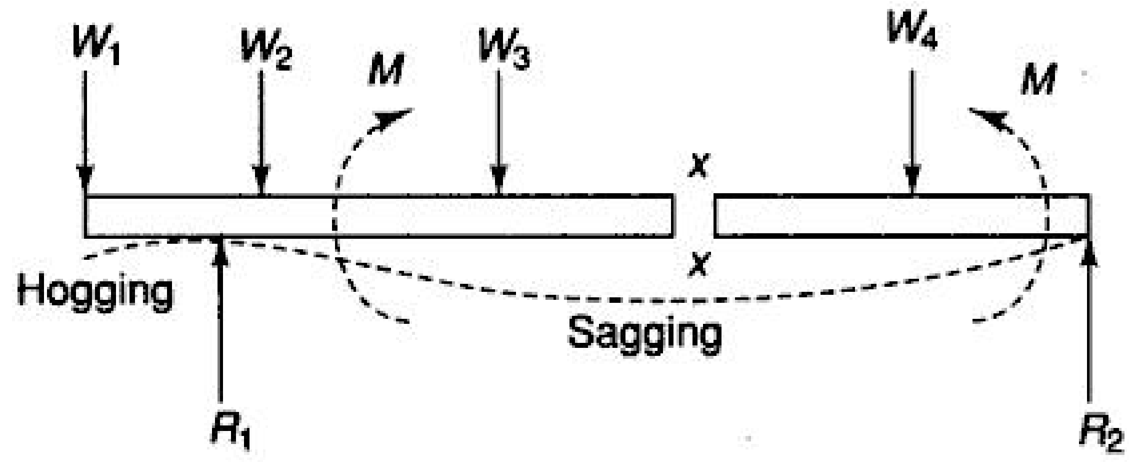

Fig. 4.4

convexity upwards is taken as negative bending moment and is called *hogging* bending moment.

A *bending moment diagram (BMD)* shows the variation of bending moment along the length of a beam.

4.5 RELATION BETWEEN *W*, *F* AND *M*

Consider a small length δx cut out from a loaded beam at a distance x from a fixed origin O (Fig. 4.5). Let

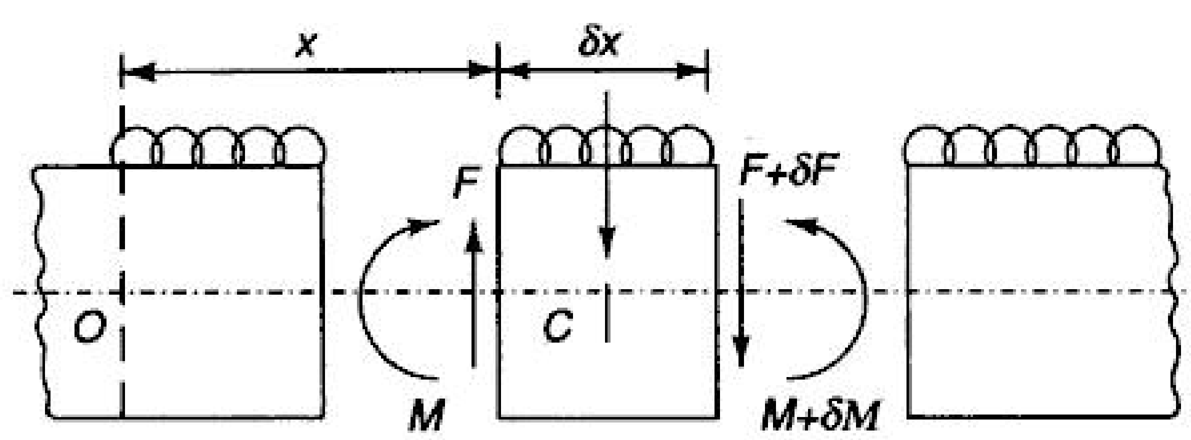

Fig. 4.5

w = mean rate of loading on the length δx
F = shear force at the section x
$F + \delta F$ = shear force at the section $x + \delta x$
M = bending moment at the section x
$M + \delta M$ = bending moment at the section $x + \delta x$

Total load on the length $\delta x = w.\delta x$ acting approximately through the centre C (if the load is uniformly distributed, it will be exactly acting through C).

For equilibrium of the element of length δx, equating vertical forces,

$$F = w\delta x + (F + \delta F) \quad \text{or} \quad w = -\frac{dF}{dx} \tag{4.1}$$

that is, rate of change of shear force (or slope of the shear force curve) is equal to intensity of loading.

Taking moments about C, $M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} - (M + \delta M) = 0$

Neglecting the product and squares of small quantities,

$$F = \frac{dM}{dx} \tag{4.2}$$

i.e. rate of change of bending moment is equal to the shear force.

- Portion EG: $F_x = 4$ kN (constant);
- Portion DE: $F_x = 4 + 2(x - 4)$ (linear); $F_e = 4$ kN; $F_d = 10$ kN
- Portion CD: $F_x = 4 + 2(x - 4) + 6$; (linear); $F_d = 16$ kN; $F_c = 22$ kN
- Portion BC: $F_x = 22 + 4$; (constant); $F_c = F_b = 26$ kN;
- Portion AB: $F_x = 26 + 6$; (constant); $F_a = F_b = 32$ kN;

Shear force diagram has been shown in Fig. 4.10*b*.

Bending moment diagram

- Portion EG: $M_x = 4x$ (linear); $M_g = 0$; $M_e = 16$ kN.m
- Portion DE: $M_x = 4x + \dfrac{2(x-4)^2}{2}$ ...(parabolic); $M_e = 16$ kN.m ; $M_d = 37$ kN.m

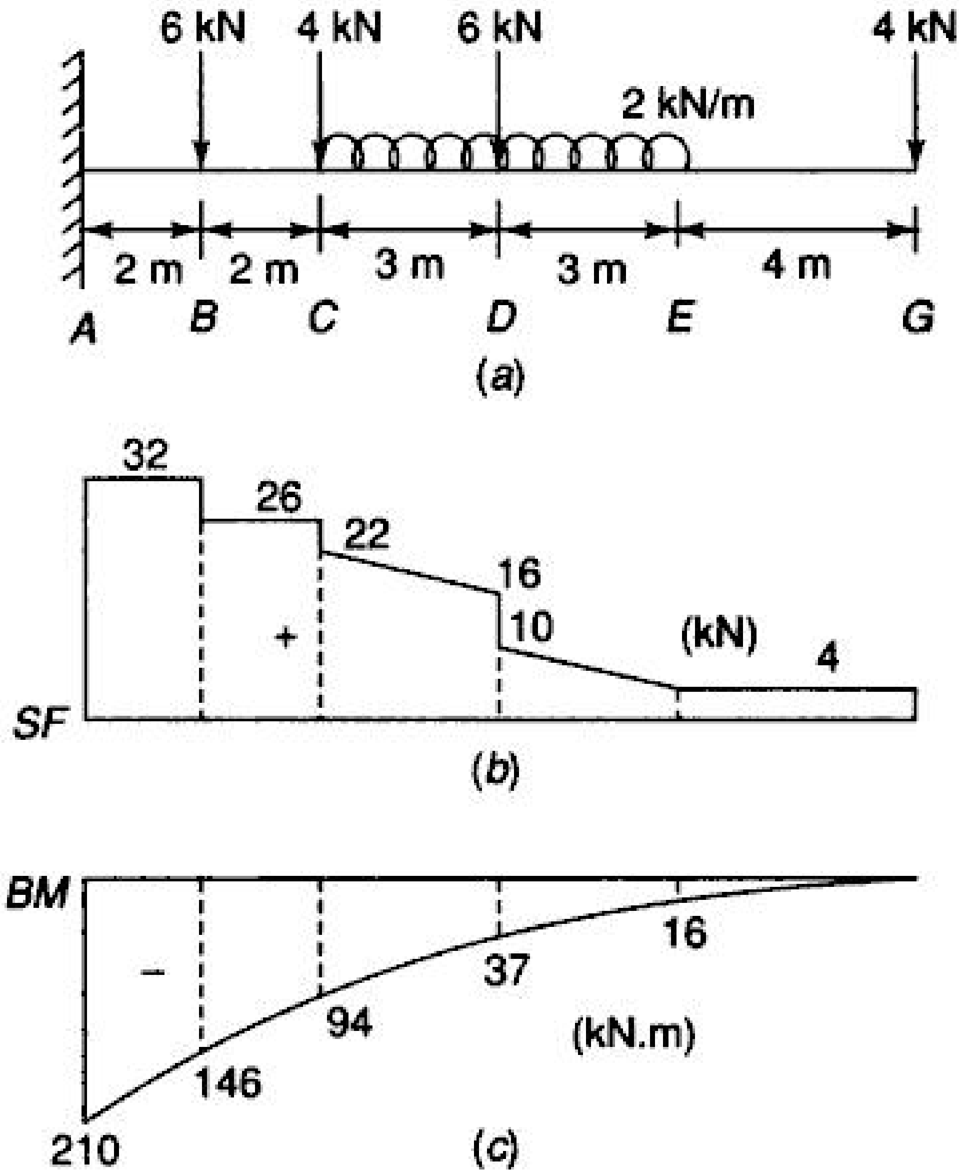

Fig. 4.10

- Portion CD: $M_x = 4x + \dfrac{2(x-4)^2}{2} + 6(x - 7)$ (parabolic)

  $M_{d(x=7)} = 37$ kN.m; $M_{c(x=10)} = 94$ kN.m
- Portion BC: $M_x = 4x + 2 \times 6(x - 7) + 6(x - 7) + 4(x - 10)$ (linear)

  $M_{c(x=10)} = 94$ kN.m; $M_{b(x=12)} = 146$ kN;
- Portion AB: $M_x = 4x + 2 \times 6(x - 7) + 6(x - 7) + 4(x - 10) + 6(x - 12)$ (linear)

  $M_{b(x=12)} = 146$ kN.m; $M_{a(x=14)} = 210$ kN

Bending moment diagram has been shown in Fig. 4.10*c*.

Example 4.4 *A cantilever is loaded with distributed load of varying intensity with zero load at the free end as shown in Fig. 4.11a. Draw the shear force and bending moment diagrams.*

Solution

Intensity of loading at any cross-section C at a distance x from free end $= \dfrac{w}{l}.x$

Shear force diagram

At a distance x from B, $F_x = \dfrac{1}{2}\dfrac{wx}{l}.x = \dfrac{wx^2}{2l}$ (parabolic); $F_b = 0$; $F_a = \dfrac{wl}{2}$

Shear force diagram is shown in Fig. 4.11*b*.

Bending moment diagram

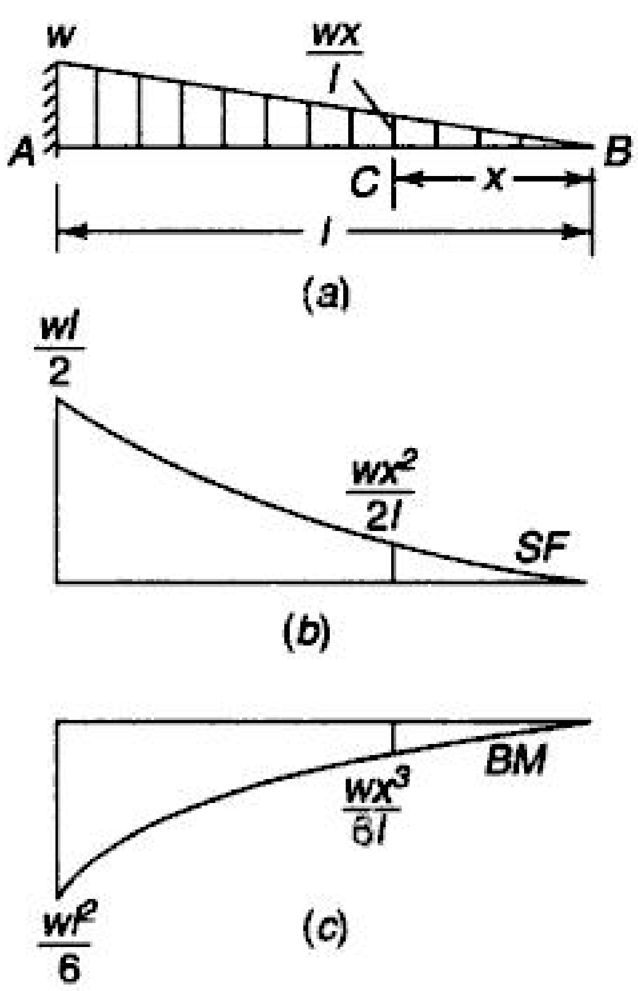

Fig. 4.11

Bending moment diagram

The bending moment at a section is found by treating the distributed load as acting at its centre of gravity.

$$M_x = R_a.x - wx.\frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(l - x) \qquad \text{(parabolic)}$$

$$M_{a(x=0)} = 0; M_{b(x=l)} = 0;$$

For maximum value, $F = \frac{dM}{dx} = 0$ or $\frac{wl}{2} - wx = 0$ or $x = l/2$

Thus maximum bending moment, $M_{(x=l/2)} = \frac{wl^2}{8}$

Bending moment diagram is shown in Fig. 4.15*c*.

Uniformly Distributed Load with Equal Overhangs

Let w be the uniformly distributed load on the beam as shown in Fig. 4.16*a*.

As the overhangs are equal, $R_a = R_b = \frac{w(l+2a)}{2}$

Shear force diagram

- Portion *DA*: $F_x = -wx$ (linear); $F_d = 0$; $F_a = -wa$
- Portion *AB*: $F_x = -wx + \frac{w(l+2a)}{2}$ (linear); $F_{a(x=a)} = \frac{wl}{2}$; $F_{b(x=l+a)} = -\frac{wl}{2}$
- Portion *BE*: $F_x = -wx + \frac{w(l+2a)}{2} + \frac{w(l+2a)}{2} = -wx + w(l+2a)$ (linear)

$$F_{b(x=l+a)} = wa; \; F_{e(x=l+2a)} = 0;$$

Shear force diagram is shown in Fig. 4.16*b*.

Bending moment diagram

- Portion *DA*: $M_x = -\frac{wx^2}{2}$ (parabolic) ; $M_d = 0$; $M_a = -\frac{wa^2}{2}$
- Portion *AB*: $M_x = -\frac{wx^2}{2} + \frac{w(l+2a)}{2}(x - a)$ (parabolic)

$$M_a = -\frac{wa^2}{2}; M_{b(x=l+a)} = -\frac{wa^2}{2}$$

- Portion *BE*: Bending moment will be reducing to zero in a parabolic manner at *E*. It is convenient to consider it from end *E*. Then $M_x = -wx^2/2$.

At midpoint *C*,

$$M_{c(x=a+l/2)} = -\frac{w(a+l/2)^2}{2} + \frac{w(l+2a)}{2}.\frac{l}{2}$$

$$= -\frac{w}{2}\left(a^2 + \frac{l^2}{4} + al - \frac{l^2}{2} - al\right) = \frac{w}{8}(l^2 - 4a^2)$$

It is maximum when $dM/dx = 0$ or $-wx + \frac{5w(2+a)}{3} = 0$

or $-3x + 5(2 + a) = 0$ or $x = 5(2 + a)/3$

$$\therefore \text{Maximum bending moment} = -\frac{w[(5/3)(2+a)]^2}{2} + \frac{5w(2+a)}{3}\left(\frac{5}{3}(2+a) - a\right)$$

$$= -\frac{25w}{18}(2+a)^2 + \frac{5w(2+a)}{9}(10+2a)$$

$$= \frac{5w}{18}[(2+a)(-10-5a+20+4a)]$$

$$= \frac{5w}{18}[(2+a)(10-a)]$$

$$= \frac{5w}{18}(20+8a-a^2)$$

The maximum bending moment will be as small as possible if the magnitudes of the sagging and the hogging bending moments are equal. Thus equating the positive and negative bending moments,

$$\frac{5w}{18}(20+8a-a^2) = \frac{wa^2}{2}$$

or $5(20 + 8a - a^2) = 9a^2$ or $14a^2 - 40a - 100 = 0$

or $$a = \frac{40 \pm \sqrt{1600 + 4 \times 14 \times 100}}{28} = \frac{40 \pm 84.85}{28}$$

As negative value of a is not practical, taking positive sign, $a = 4.46$ m

Thus distance of piers from the ends = 4.46 m and (8 – 4.46) = 3.54 m

$$R_a = \frac{5w(2+a)}{3} = \frac{5w(2+4.46)}{3} = 10.77w$$

and $R_b = 20w - 10.77w = 9.23w$

Bending moment diagram

$$M_{max} = \frac{wa^2}{2} = \frac{w \times 4.46^2}{2} = 9.45w \text{ at } A$$

and at $$x = \frac{5}{3}(2+a) = \frac{5}{3}(2+4.46) = 10.77 \text{ m}$$

- Portion AB: $M_x = -\frac{wx^2}{2} + 10.77w(x-4.46)$ (parabolic);

  $M_{b(x = 4.46 + 12)} = -6.23w$

- Portion BE: $M_x = -\frac{wx^2}{2} + 10.77w(x-4.46) + 9.23w(x-16.46)$ (parabolic)

  $M_{e(x=20)} = 0$; $M_b(x = 4.46 + 12) = -6.23w$

M_b can also be considered from end E, $M_b = -\frac{(8-4.46)^2 w}{2} = -6.26w$

- Portion *BC*: *x* from end *C*.

$$M_x = -\frac{x^2}{2}\frac{x}{3} = -\frac{x^3}{6} \text{ ...(cubic); } F_c = 0;\ F_b = -288 \text{ kN.m}$$

Bending moment diagram is shown in Fig. 4.21*c*.

Example 4.11 *A simply supported beam has distributed load of varying intensity with zero at one end to w per unit run at the other. Draw the shear force and bending moment diagrams.*

Solution The loading on the beam is shown in Fig. 4.22*a*.

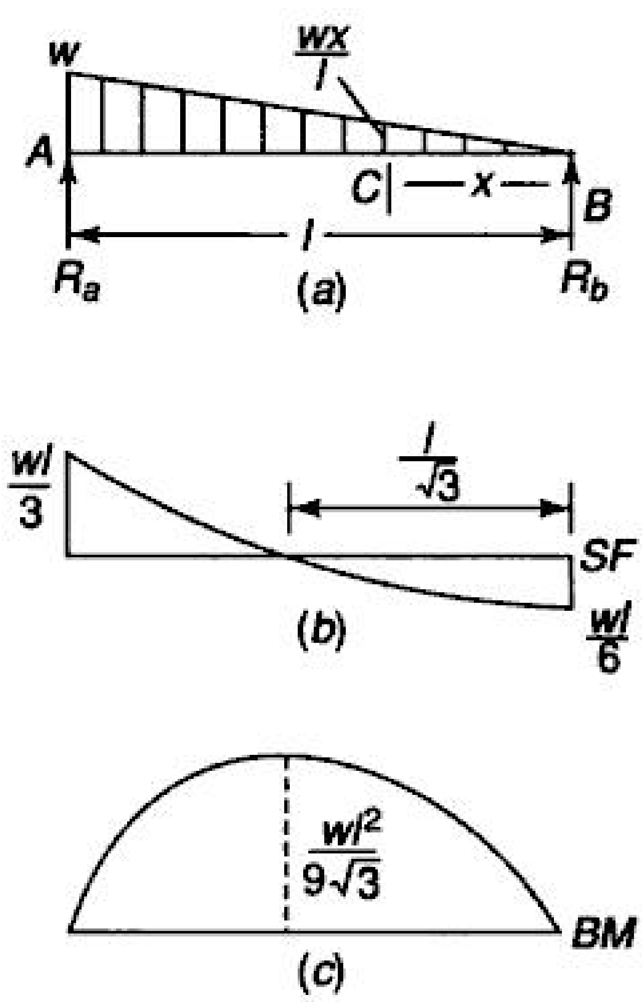

Fig. 4.22

Taking moments about *B*, $R_a \times l = \frac{wl}{2}.\frac{2l}{3}$

or $R_a = \frac{wl}{3}$ and $R_b = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$

Intensity of loading at a distance *x* from end *B*

$$= \frac{w}{l}x$$

$$F_x = -R_b + \frac{1}{2}\frac{wx}{l}x = -\frac{wl}{6} + \frac{wx^2}{2l} \text{ (parabolic);}$$

$F_b = -wl/6;\ F_a = wl/3$

Shear force is zero at $-\frac{wl}{6} + \frac{wx^2}{2l} = 0$ or $x = \frac{l}{\sqrt{3}}$

$$M_x = \frac{wl}{6}x - \frac{wx^2}{2l}\frac{x}{3} = \frac{wlx}{6} - \frac{wx^3}{6l} \text{ (cubic); } M_b = 0;\ M_a = 0$$

Maximum bending moment is at $x = \frac{l}{\sqrt{3}}$, $M_{max} = \frac{wl^2}{6\sqrt{3}} - \frac{wl^2}{6 \times 3\sqrt{3}} = \frac{wl^2}{9\sqrt{3}}$

Shear force and bending moment diagrams have been shown in Fig. 4.22*b* and *c* respectively.

Example 4.12 *A beam of 9-m span supports a 160-mm thick concrete wall. The height of the wall is 1 m at the left end and increases to 2 m at the right end. The beam has two supports, one at 2 m from the left end and the other at 1 m from the right end. Find the maximum bending moment on the beam if the concrete weighed 25 kN/m³. Draw the shear force and bending moment diagrams.*

Solution The loading on the beam is shown in Fig. 4.23*a*.

Intensity of load at *C* = Volume per m length × 25 = $(1 \times 1 \times 0.16) \times 25$ = 4 kN/m

Intensity of load at *D* = $(1 \times 2 \times 0.16) \times 25$ = 8 N/m

The loading can be divided into

(i) uniformly distributed load of 4 kN/m and

Solution Taking moments about B,

$$R_a \times 10 = 12 \times 5 + (2 \times 8) \times 0$$

or $R_a = 6$ kN

and $R_b = 12 + 16 - 6 = 22$ kN

The effect of the bracket is to apply a load of 12 kN and a bending moment of (12 × 2) kN.m at the point C (Fig. 4.26*b*).

Shear force diagram

- Portion AC: $F_x = 6$ kN (constant)
- Portion CD: $F_x = 6 - 12 = -6$ kN (constant)
- Portion DB: $F_x = -6 - 2(x - 6)$ (linear)
  $F_{d(x=6)} = -6$ kN; $F_{b(x=10)} = -14$ kN
- Portion BE: $F_x = -6 - 2(x - 6) + 22$ (linear)
  $F_{b(x=10)} = 8$ kN; $F_{e(x=14)} = 0$
  Shear force diagram is shown in Fig. 4.26*c*.

Bending moment diagram

- Portion AC: $M_x = 6x$ (linear)
  $M_{a(x=0)} = 0$; $M_{c(x=3)} = 18$ kN.m
- Portion CD: $M_x = 6x - 12(x - 3) + 24$ (linear)
  $M_{c(x=3)} = 42$ kN; $M_{d(x=6)} = 24$ kN.m
- Portion DB:

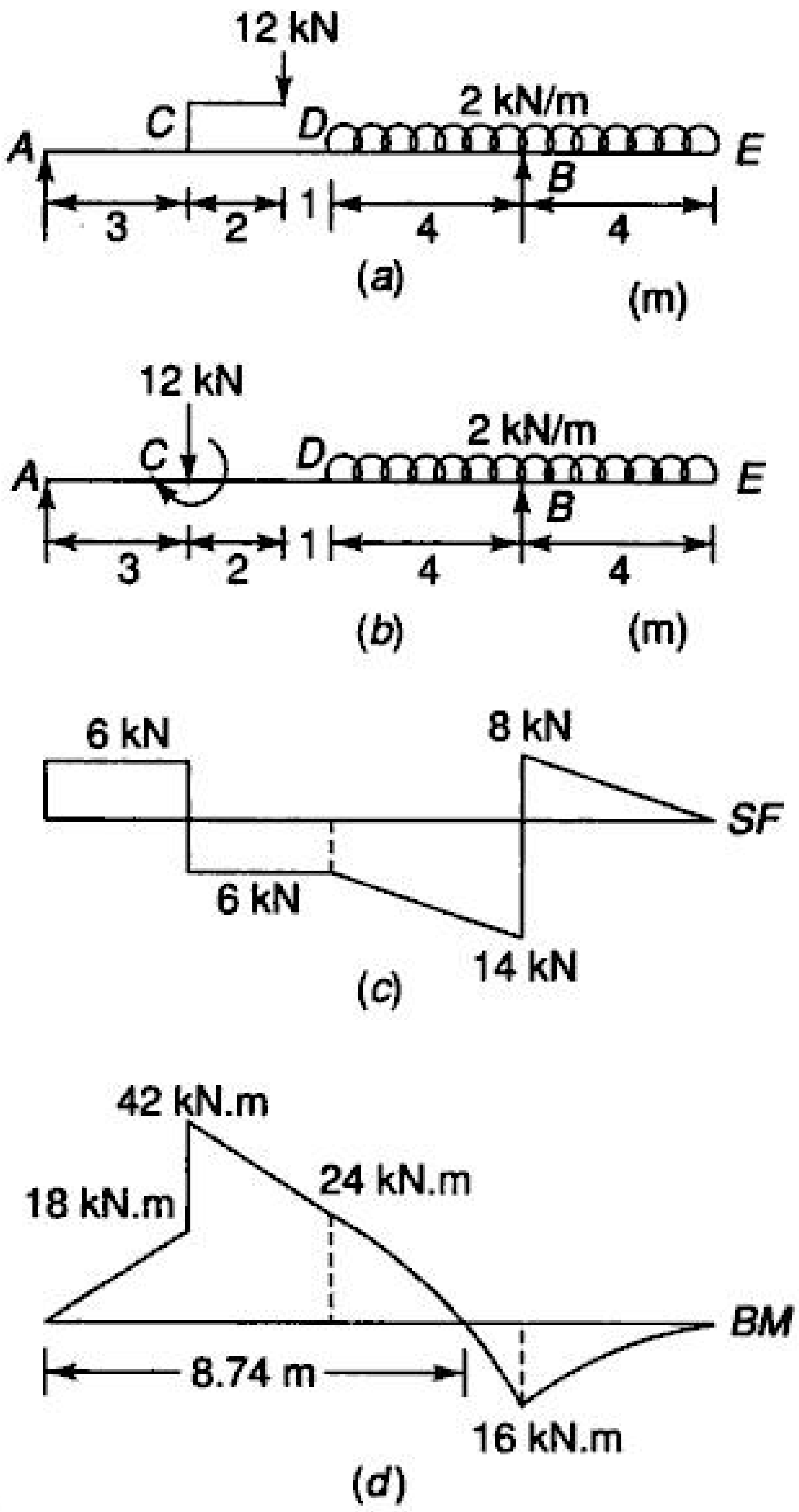

Fig. 4.26

$$M_x = 6x - 12(x - 3) + 24 - \frac{2(x-6)^2}{2} \quad \text{(parabolic)}$$

$M_{d(x=6)} = 24$ kN; $M_{b(x=10)} = -16$ kN

It is zero at $6x - 12(x-3) + 24 - \dfrac{2(x-6)^2}{2} = 0$ or $-6x + 60 - x^2 - 36 + 12x = 0$

or $x^2 - 6x - 24 = 0$ or $x = \dfrac{6 \pm \sqrt{36 + 4 \times 24}}{2} = 3 \pm 5.74 = 8.74$ m

(taking positive value)

- Portion BE: from end B, $M_x = -\dfrac{2x^2}{2} = -x^2$ (parabolic)
  $F_{b(x=4)} = -16$ kN; $F_{e(x=0)} = 0$

Bending moment diagram is shown in Fig. 4.26*d*.

Example 4.16 *Draw shear force and bending moment diagrams for the beam shown in Fig. 4.27a.*

Solution There is no bending moment at point C.

Taking moments about C for the left portion,

$$R_a \times 6 = \left(\frac{1}{2} \times 8 \times 6\right)\left(\frac{2}{3} \times 6\right) \quad \text{or} \quad R_a = 16 \text{ kN}$$

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- There is an upward force of (10.5 + 6 = 16.5 kN) at C which indicates the reaction of the support.
- There is drop of 6 kN at D indicating a point load of 6 kN.

Load diagram is shown in Fig. 4.29*b*.

Bending moment diagram

Portion AB: $M_x = 7.5x - \dfrac{1 \times x^2}{2}$ (parabolic); $M_a = 0$; $M_{b(x=4)} = 22$ kN.m

Portion BC: $M_x = 7.5x - \dfrac{x^2}{2} - 9(x-4)$ (parabolic);

$$M_{b(x=4)} = 22 \text{ kN.m}; \; M_{c(x=9)} = -18 \text{ kN.m}$$

It is zero when $7.5x - \dfrac{x^2}{2} - 9(x-4) = 0$ or $x = 7.12$ m

Portion CD: $M_x = 7.5x - 9(x-4.5) - 9(x-4) + 16.5(x-9)$ (linear)

$$M_{c(x=9)} = -18 \text{ kN.m}; \; M_{d(x=12)} = 0$$

Bending moment diagram is shown in Fig. 4.29*c*.

Summary

- A structural element which is subjected to loads transverse to its axis is known as a *beam.*
- A beam with both of its ends on simple supports is known as a *simply supported beam.* Each support exerts a reaction on the beam.
- A beam with one end fixed and the other end free is called a *cantilever.* There is a vertical reaction and moment at the fixed end (known as *fixing moment*).
- Generally, beams with more than two reaction components cannot be analysed using the equations of static equilibrium alone and are known as *statically indeterminate beams.*
- *Shear force* is the unbalanced vertical force on one side (to the left or right) of a section of a beam and is the sum of all the normal forces on one side of the section.
- Shear force is considered positive when the resultant of the forces to the left of a section is upwards or to the right downwards.
- A *shear force diagram* shows the variation of shear force along the length of a beam.
- *Bending moment* at a section of a beam is the algebraic sum of the moments about the section of all the forces on one side of the section.
- The bending moment causing concavity upwards is referred as *sagging* bending moment and is taken as positive. A bending moment causing convexity upwards is taken as negative and is called *hogging* bending moment.
- A *bending moment diagram* shows the variation of bending moment along the length of a beam.
- Rate of change of shear force (or slope of the shear force curve) is equal to intensity of loading.

5.2 THEORY OF SIMPLE BENDING

The following theory is applicable to the beams subjected to simple or pure bending when the cross-section is not subjected to a shear force since that will cause a distortion of the transverse planes. The assumptions being made are as under:

(i) The material is homogeneous and isotropic, i.e. it has the same values of Young's modulus in tension and compression.

(ii) Transverse planes remain plane and perpendicular to the neutral surface after bending.

(iii) Initially the beam is straight and all longitudinal filaments are bent into circular arcs with a common centre of curvature which is large compared to the dimensions of the cross-section.

(iv) The beam is symmetrical about a vertical longitudinal plane passing through vertical axis of symmetry for horizontal beams.

(v) The stress is purely longitudinal and the stress concentration effects near the concentrated loads are neglected.

Consider a length of beam under the action of a bending moment M as shown in Fig. 5.2*a*. NN is considered as the original length of the beam. The neutral surface is a plane through XX. In the side view NA indicates the neutral axis. O is the centre of curvature on bending (Fig. 5.2*b*).

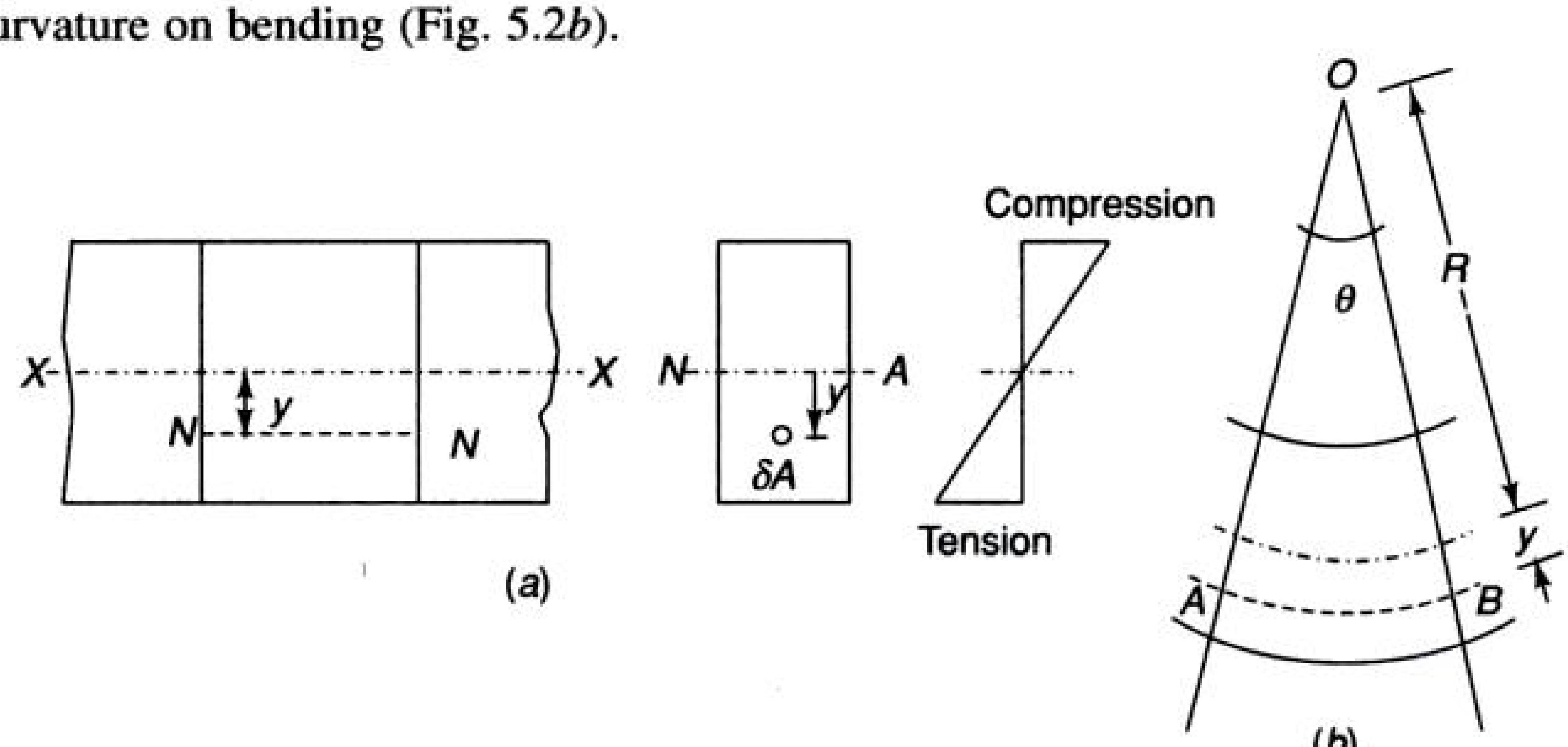

Fig. 5.2

Let R = radius of curvature of the neutral surface

θ = angle subtended by the beam length at centre O

σ = longitudinal stress

A filament of original length NN at a distance y from the neutral axis will be elongated to a length AB

The strain in $AB = \dfrac{AB - NN}{NN}$ (original length of filament AB is NN)

or
$$\frac{\sigma}{E} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

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or $\sigma.\frac{15\times15^2}{6} = \frac{360\times800}{4}$ or σ = 128 MPa

Now, in the cantilever let the loading be w N per m run to break it.

Maximum bending moment = $\frac{wl^2}{2} = \frac{w\times1.6^2}{2} = 1.28w$ N.m

(In the above relation as w is in N/m, l has to be in consistent units, i.e. in m)

Moment of resistance $= \sigma.Z = \sigma.\frac{bd^2}{6} = 128\times\frac{40\times75^2}{6}$

$= 4800 \times 10^3$ N.mm or 4800 N.m

Equating the two, $1.28w = 4800$ or $w = 3750$ N/m

Example 5.2 *A floor carries a load of 8 kN/m² and is supported by joists 120-mm wide and 240-mm deep over a span of 6 m. Determine the spacing centre to centre of the joists if the maximum allowable bending stress is 10 MPa.*

Solution Let the spacing of the joists be s m (Fig. 5.8),

Loading on the joist per unit length

w = Area supported by joist per unit length × Load/unit area

= (Spacing of the joists × 1) × Load/unit area

= $s \times 8000$

= 8000 s N/m

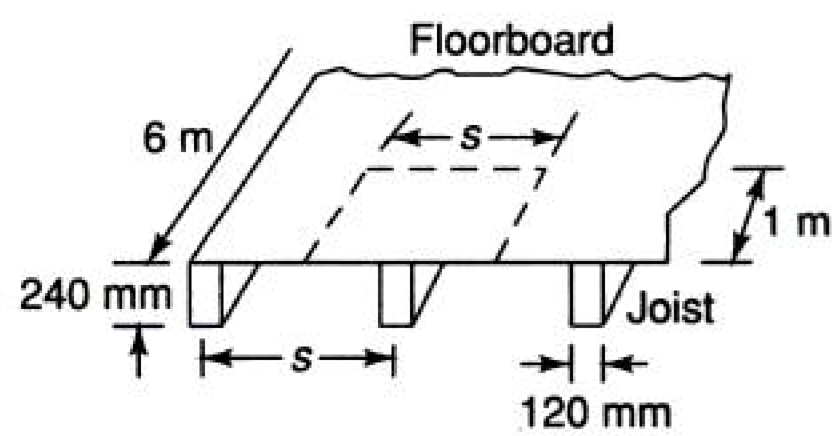


Fig. 5.8

Maximum bending moment $= \frac{wl^2}{8}$

$= \frac{8000s\times6^2}{8}$

$= 36\,000s$ N.m

∴ Moment of resistance $= \sigma.\frac{bd^2}{6} = 10\times\frac{120\times240^2}{6}$

$= 11\,520 \times 10^3$ N.mm or 11 520 N.m

Equating the two, $36\,000s = 11\,520$

or $s = 0.32$ m or 320 mm

Example 5.3 *A simply supported beam of 6.75-m span is made up of symmetrical I-section (Fig. 5.9a). Determine what concentrated load can be carried at a distance of 2.25 m from one support if the maximum permissible stress is 80 MPa.*

Solution Let W kN be the concentrated load so that the reaction at the supports are $W/3$ and $2W/3$ as shown in Fig. 5.9b.

Maximum bending moment

Similarly for the inner hexagon, $I_{xx} = 0.541b^4$

$$B = 80 \text{ mm};\ h = 80 \sin 60^\circ = 80 \times 0.866 = 69.28 \text{ mm}$$

$$b = B - 2(5\tan 30^\circ) = 80 - 5.77 = 74.23 \text{ mm}$$

Thus for the hollow hexagonal tube

$$I_{xx} = 0.541(B^4 - b^4) = 0.541(80^4 - 74.23^4) = 5734 \times 10^6 \text{ mm}^4$$

$$Z_{xx} = \frac{I_{xx}}{h} = \frac{5734 \times 10^6}{69.28} = 82\,766 \text{ mm}^3$$

Figure 5.5*c* shows the loading on the beam. Let *a* m be the distance of each load from the respective end support.

Each reaction = 24 kN

Maximum bending moment (at the centre)

$$= 24 \times \frac{2.4}{2} - 24 \times \left(\frac{2.4}{2} - a\right) = 28.8 - 28.8 + 24a$$

$$= 24a \text{ kN.m or } 24a \times 10^6 \text{ N.mm}$$

Now, $M = \sigma.Z$

or $24a \times 10^6 = 120 \times 82\,766$ or $a = 0.414$ m

Least distance between the loads = 2.4 – 2 × 0.414 = 1.572 m

Example 5.7 *Figure 5.13a shows the section of a beam. Determine the ratio of its moment of resistance to bending in the y-plane to that in the x-plane if the maximum bending stress remains same in the two cases.*

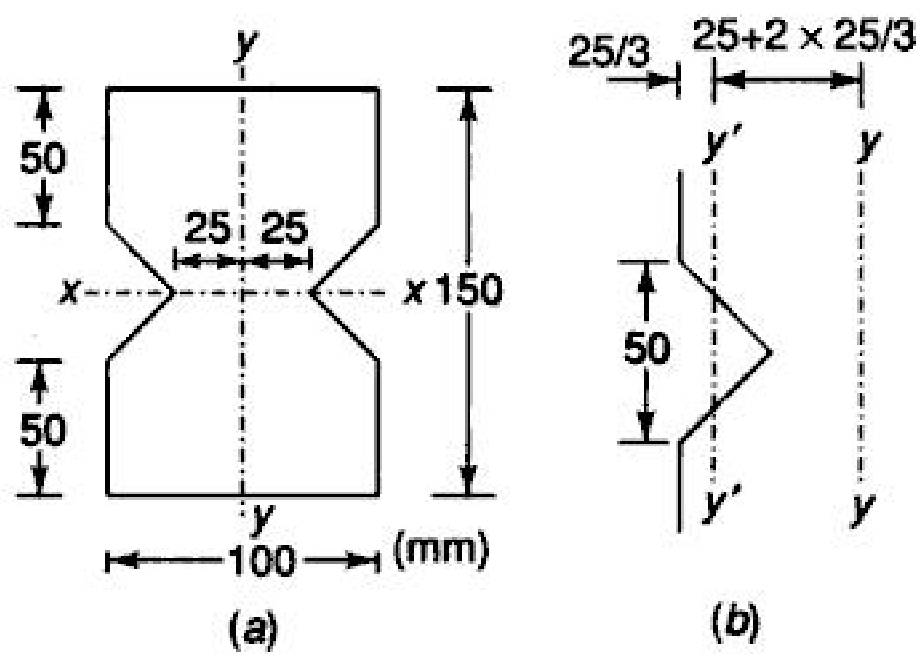


Fig. 5.13

Solution

$$I_{xx} = \frac{1}{12} \times 100 \times 150^3 - 4 \times \frac{25 \times 25^3}{12}$$

$$= 27.995 \times 10^4 \text{ mm}^4$$

$$Z_{xx} = \frac{I_{xx}}{y_{max}} = \frac{27.995 \times 10^6}{75} = 0.373 \times 10^6 \text{ mm}^3$$

$$I_{yy} = \frac{1}{12} \times 150 \times 100^3 - 2\left[\frac{50 \times 25^3}{36} + \frac{50 \times 25}{2}\left(25 + \frac{50}{3}\right)^2\right] \text{ ... (refer Fig. 5.13b)}$$

$$= 12.5 \times 10^6 - 2.214 \times 10^6 = 10.286 \times 10^6 \text{ mm}^4$$

$$Z_{yy} = \frac{10.286 \times 10^6}{50} = 0.2057 \times 10^6 \text{ mm}^3$$

$$\frac{M_{yy}}{M_{xx}} = \frac{Z_{yy}}{Z_{xx}} = \frac{0.2057 \times 10^6}{0.373 \times 10^6} = 0.552$$

5.4 BEAMS WITH UNIFORM BENDING STRENGTH

Usually, the beams are designed on the basis of maximum bending stress occurring at any cross-section of the beam and a constant cross-section is provided throughout the length of the beam. However, as the actual moment and thus the stress varies and is less at all other cross-sections along the length of the beam, a beam with constant cross-section or with uniform moment of resistance is uneconomical. In beams with heavy loads, beams may be designed on the basis of variation in the bending moment. Such beams will have the same maximum bending stress all along the length and are known as *beams with uniform bending strength.* This can be achieved by having either a uniform width of the section or a uniform depth.

Beam with Constant Width and of Varying Depth

Let l be the length and b the constant width of a beam with uniform strength. Also let the depth be d_x at a distance x from the support.

Then, moment of resistance of the section, $M_r = \sigma Z_x = \sigma \dfrac{bd_x^2}{6}$

Moment of resistance of the section will depend upon the loading on the beam.

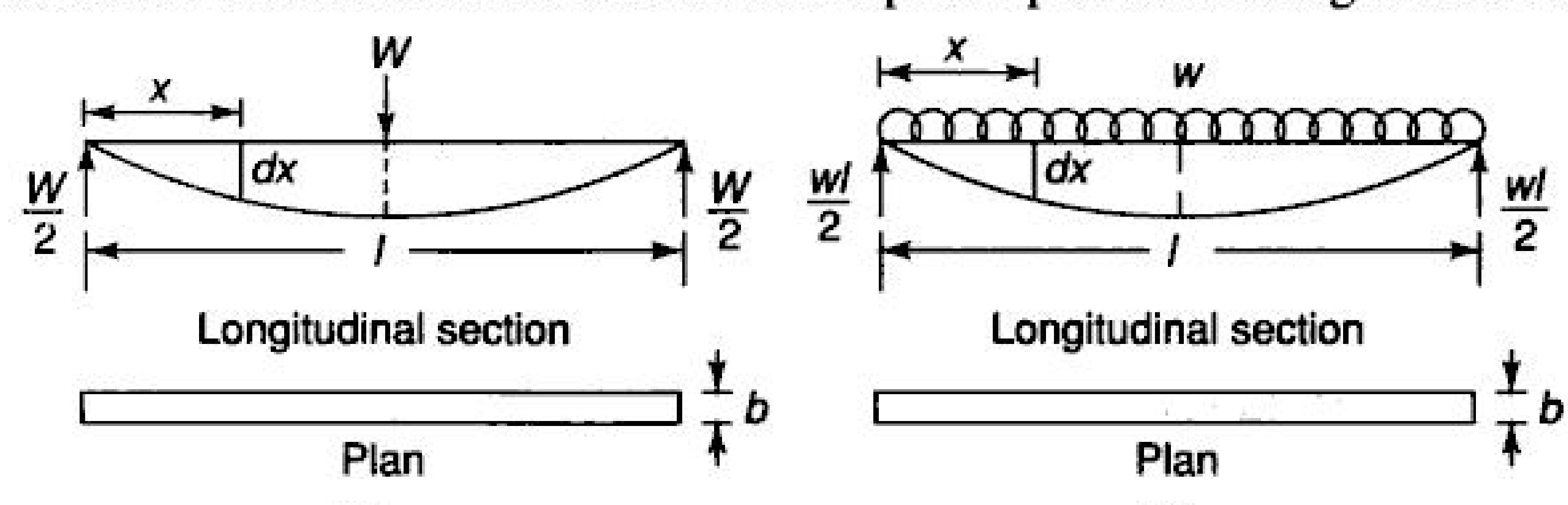


Fig. 5.18

(i) *Concentrated load W at the midspan* (Fig. 5.18*a*)

Bending moment at the section = $\dfrac{W}{2}x$

Equating the moment of resistance and the bending moment,

$$\sigma \frac{bd_x^2}{6} = \frac{W}{2}x \quad \text{or} \quad d_x^2 = \frac{3W}{\sigma b}x \quad \text{or} \quad d_x = \sqrt{\frac{3W}{\sigma b}.x} = k\sqrt{x}$$

where k is a constant. The expression indicates that the variation is parabolic.

At the centre $d_x = k\sqrt{l/2}$

(ii) *Uniformly distributed load throughout* (Fig. 5.18*b*)

Bending moment at the section = $\dfrac{wl}{2}x - \dfrac{wx^2}{2} = \dfrac{wx}{2}(l - x)$

Equating the moment of resistance and the bending moment,

$$\sigma \frac{bd_x^2}{6} = \frac{wx}{2}(l - x) \quad \text{or} \quad d_x^2 = \frac{3wx(l - x)}{\sigma b} \quad \text{or} \quad d_x = \sqrt{\frac{3wx(l - x)}{\sigma b}}$$

If tensile stress reaches to maximum value of 18 MPa,

Compressive stress

$$= 18 \times \frac{280 - 89.13}{89.13} = 38.6 \text{ MPa}$$

which is well within permissible limits.

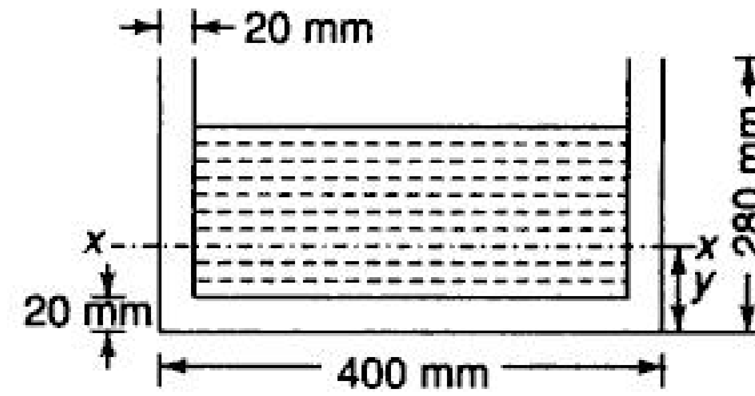


Fig. 5.22

Thus moment of resistance

$$= \frac{18 \times 147.48 \times 10^6}{89.13} = 29.784 \times 10^6 \text{ N.mm or } 29\,784 \text{ N.m}$$

Let w be the total weight (channel + water) N per m length of the channel.

Then $\frac{w \times 11^2}{8} = 29\,784$ or $w = 1969.2$ N/m

Let x m be the level of water in the channel.

Weight of channel + weight of water = Total weight/m

$$(0.36 \times 0.02 + 2 \times 0.28 \times 0.02) \times 68\,000 + 0.36x \times 9810 = 1969.2$$

$$1251.2 + 3531.6\,x = 1969.2$$

or $x = 0.203$ m or 203 mm

Example 5.16 *The cross-section of a beam is shown in Fig. 5.23a. Determine the moment of resistance of the section about the horizontal neutral axis for both positive and negative bending moment. The permissible stresses in tension and compression are 24 MPa and 85 MPa respectively.*

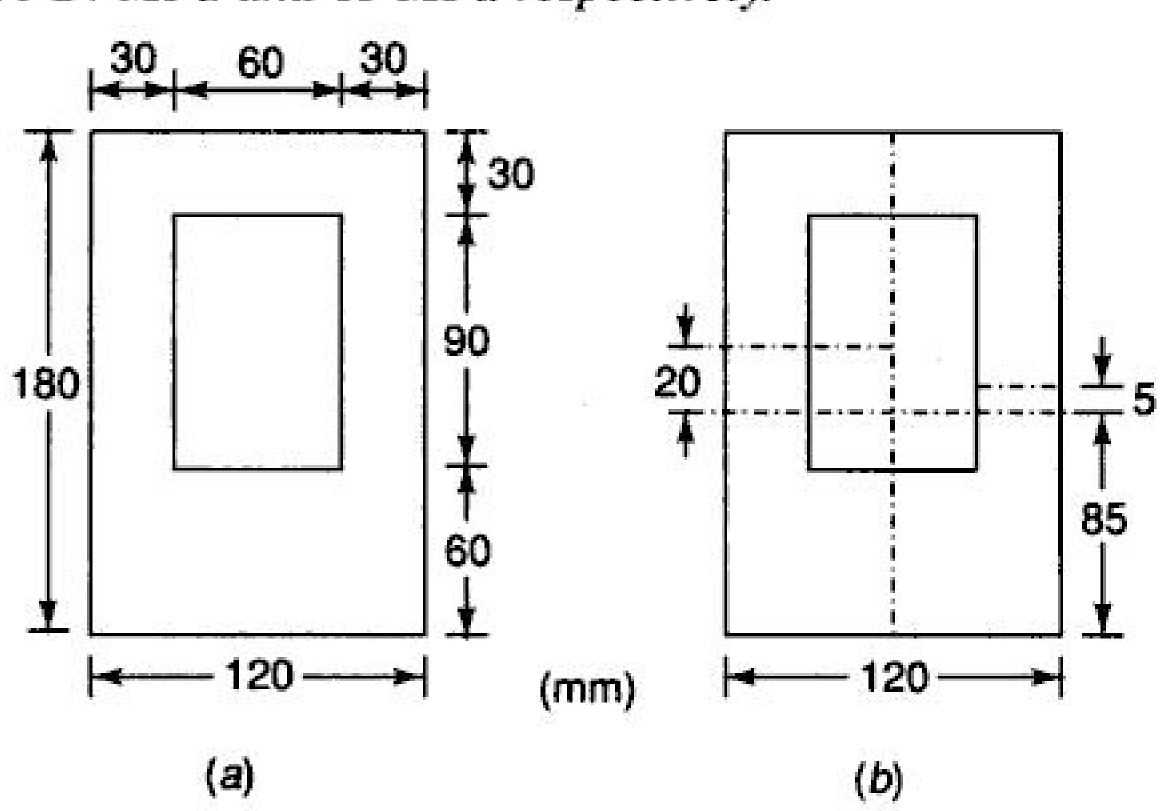


Fig. 5.23

Solution Refer Fig. 5.23*b*,

Distance of the centroid of the net section from the bottom edge,

$$y = \frac{120 \times 180 \times 90 - 60 \times 90 \times (60 + 45)}{120 \times 180 - 60 \times 90} = 85 \text{ mm}$$

$$I = \left[\frac{120 \times 180^3}{12} + 120 \times 180 \times 5^2\right] - \left[\frac{60 \times 90^3}{12} + 60 \times 90 \times 20^2\right]$$

$$= 58.86 \times 10^6 - 5.805 \times 10^6 = 53.055 \times 10^6 \text{ mm}^4$$

Example 5.20 *Two rectangular bars, one of brass and the other of steel, each 36 mm by 9 mm are placed together to form a beam of 36-mm width and 18-mm depth on two supports 800 mm apart, the steel bar being at the bottom. Find the maximum central load if the bars are*

(i) separate and can bend independently

(ii) firmly secured to each other throughout the length.

Maximum permissible stress in steel is 102 MPa and in brass 72 MPa. Take E_s *= 204 GPa and* E_b *= 85 GPa.*

Solution Refer Fig. 5.27,

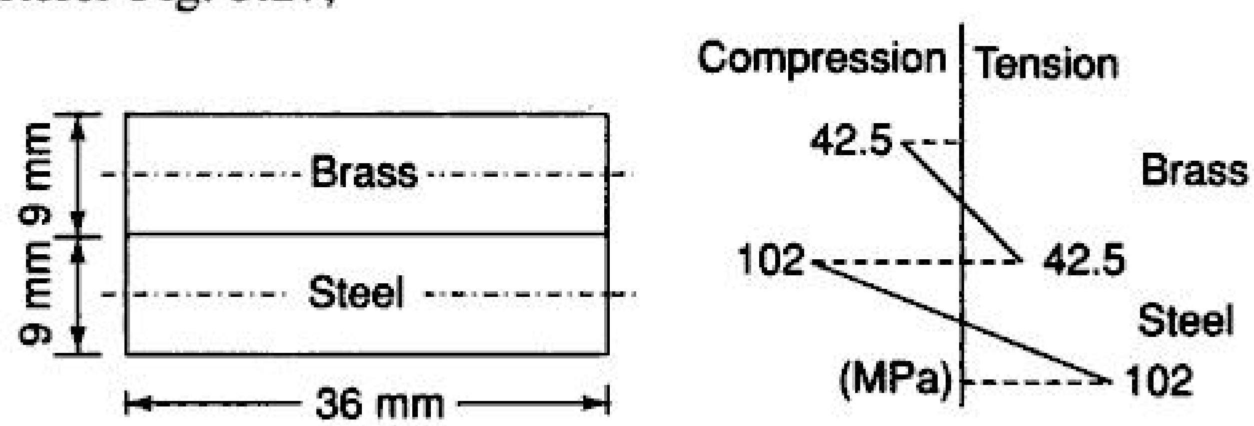


Fig. 5.27

(i) When the bars are separate and can bend independently, each will have its own neutral axis.

We have $$\frac{\sigma}{y} = \frac{E}{R}$$

Assuming the same radius of curvature for the two bars,

$$R = \frac{y_s}{\sigma_s}E_s = \frac{y_b}{\sigma_b}E_b \text{ or } \frac{\sigma_s}{\sigma_b} = \frac{y_sE_s}{y_bE_b} = \frac{E_s}{E_b} = \frac{204}{85} = 2.4 \qquad .....(y_s = y_b)$$

If the stress in steel reaches to maximum value, the stress induced in brass

$$= 102/2.4 = 42.5 \text{ MPa}$$

$$M_r = M_{\text{steel}} + M_{\text{brass}} = 102\times\frac{36\times9^2}{6} + 42.5\times\frac{36\times9^2}{6}$$

$$= 49\,572 + 20\,655 = 70\,227 \text{ N.mm}$$

For a central load, $\frac{Wl}{4} = \frac{W\times0.8}{4} = 70\,227$ or $W = 351.1$ N

(ii) When the bars are firmly secured to each other throughout the length, they will bend about a common neutral axis *XX*. Fig. 5.28*a* shows the equivalent section in terms of brass. The dimensions of the steel parallel to the neutral axis are increased in the modular ratio 2.4.

Distance of the centroid of the net section from the bottom edge,

$$x = \frac{(36\times9)\times13.5 + (86.4\times9)\times4.5}{36\times9 + 86.4\times9} = 7.15 \text{ mm}$$

$$I = \left[\frac{86.4\times7.15^3}{3} + \frac{86.4\times1.85^3}{3}\right] + \left[\frac{36\times9^3}{12} + (36\times9)\times(18-7.15-4.5)^2\right]$$

For area (1) (2) (3)

$$= 25\,961 \text{ mm}^4$$

Height of N.A. from the bottom axis,

$$y = \frac{2\times80\times180\times90 + 200\times130\times65}{2\times80\times180 + 200\times130} = 78.14 \text{ mm}$$

$$I = \frac{2\times80\times78.14^3}{3} + \frac{2\times80\times101.86^3}{3} + \frac{200\times78.14^3}{3} + \frac{200\times51.86^3}{3}$$

For area (1) (2) (3) (4)

$= 122.93 \times 10^6 \text{ mm}^4$

Maximum tensile stress in timber (at bottom edge)

$$= \frac{10\,000\times10^3}{122.93\times10^6}\times78.14 = 6.356 \text{ MPa}$$

Maximum tensile stress in steel (at bottom edge) = $6.356\times20 = 127.13$ MPa

Maximum compressive stress in timber (at top edge)

$$= \frac{10\,000\times10^3}{122.93\times10^6}\times101.86 = 8.286 \text{ MPa}$$

or $$6.356\times\frac{101.86}{78.14} = 8.286 \text{ MPa}$$

Maximum compressive stress in timber at the level of top fiber of steel

$$= 8.286\times\frac{51.86}{101.86} = 4.219 \text{ MPa}$$

Maximum compressive stress in steel = $4.219\times20 = 84.37$ MPa

Example 5.24 *A straight bimetallic rectangular composite bar of width b and thickness 2t is made up of a strip of steel of rectangular section of width b and thickness t joined along its length by a strip of brass having the same dimensions. The bar is uniformly heated and is freely allowed to bend. Show that it bends to a radius*

$$R = \frac{E_b^2 + E_s^2 + 14E_bE_s}{12E_bE_s(\alpha_b - \alpha_s)}\cdot\frac{t}{T}$$

where α_b and α_s are the coefficients of linear expansions of brass and steel respectively and T is the rise in temperature.

Solution Let α_b is greater than α_s.

A force at the interaction of the two strips will tend to compress the brass and elongate the steel. Let this internal force be F. This force induces a direct load F at the centre of each section alongwith a bending moment in each strip (Fig. 5.33).

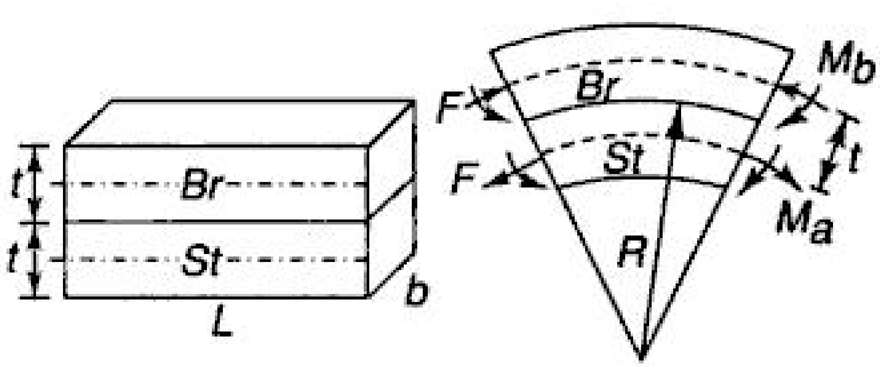


Fig. 5.33

Bending moment due to this force = $F.t$

Assuming R to be the same for both strips and much larger as compared to t,

$$M_b = \frac{I_bE_b}{R} = \frac{bt^3E_b}{12R}\text{ ; Similarly, } M_s = \frac{bt^3E_s}{12R}$$

In cases of unsymmetrical bending, if the load is applied in such a way that it passes through the *shear centre*, the above theory can still be applied by resolving the bending moment into components about the two principal axes. If a cross-section has an axis of symmetry, then it can easily be shown that this satisfies the condition for a principal axis. The other principal axis will be at right angle through the centroid.

The *shear centre* mentioned above is a point in or outside a section through which the shear force applied produces no torsion or twist of the member. If the load is not applied through the shear centre, there is twisting of the beam due to unbalanced moment caused by the shear force acting on the section. For sections symmetrical about an axis, shear centre lies on the axis of symmetry. For sections having two axes of symmetry, the shear centre lies at the intersection of these axes and thus coincides with the centroid. (Also refer section 6.5).

Example 5.27 *A 4-m long simply supported beam of 80-mm width and 100-mm depth carries a load of 10 kN at the midspan. The load is inclined at 30° to the vertical longitudinal plane and the line of action of the load passes through the centroid of the rectangular section of the beam. Determine the stresses at all the corners of the section.*

Solution The section being symmetrical, the centroid is at the centre of the rectangle and the principal axes are *x-x* and *y-y* (Fig. 5.36).

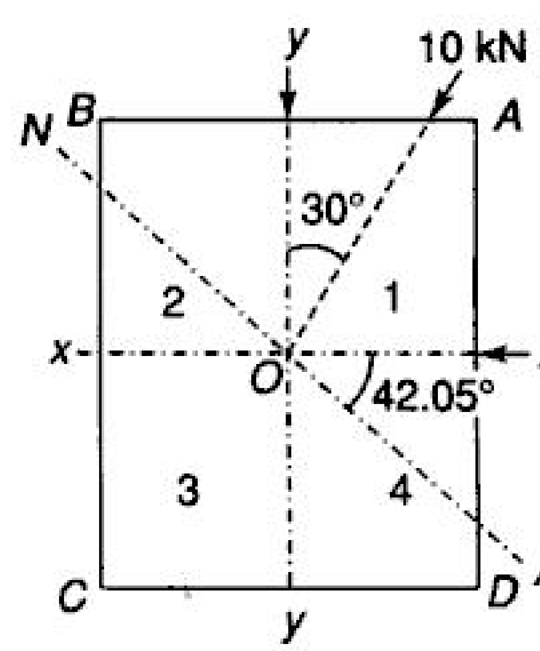


Fig. 5.36

$$I_x = \frac{80\times100^3}{12} = 6.667\times10^3 \text{ mm}^4$$

and $$I_y = \frac{100\times80^3}{12} = 4.267\times10^3 \text{ mm}^4$$

Maximum bending moment

$$= \frac{10\times4}{4} = 10 \text{ kN.m or } 10\times10^6 \text{ N.mm}$$

Resolving into components,

$M_x = 10\times10^6\cos30° = 8.66\times10^6$ N.mm (due to vertical component of load)

$M_y = 10\times10^6\sin30° = 5\times10^6$ N.mm (due to horizontal component of load)

As it is a simply supported beam,

- the vertical load component induces compressive stress in the upper half and tensile stress in the lower half and
- the horizontal load component induces compressive stress in the right half and tensile stress in the left half.

Thus the bending stress at any point (*x*, *y*) in the section consists of two parts, one due to bending about the axis *x-x* and the other due to the bending about *y-y*, i.e.

$$\sigma = \frac{M_x}{I_x}.y + \frac{M_y}{I_y}.x$$

As both the components are to give tensile stress in the 3rd quadrant, *x* and *y* both can be assumed positive in this quadrant. In the opposite quadrant, *x* and *y* both are

or $\quad -80 = -9.749\ F$ or $F = 8.206$ kN

$$\sigma_a = 0.0361 \times (-61.43) + 0.1255 \times 60$$
$$= -2.216\ F + 7.533\ F$$

or $\quad 40 = 5.317\ F$ or $F = 7.523$ kN

In quadrant containing point C, x and y are both positive,

$$\sigma_c = 0.0361 \times (180 - 61.43) + 0.1255 \times 10$$
$$= 4.278F + 1.255F$$

or $\quad 40 = 5.533\ F$ or $F = 7.229$ kN

Thus the maximum load can be 7.229 kN

5.8 DETERMINATION OF PRINCIPAL AXES

Sometimes, in case of unsymmetrical sections, the directions of the principal axes are not known. In such cases, the direction of these can be found as follows:

Let OX and OY be any two perpendicular axes through the centroid and OU and OV the principal axes (Fig. 5.40). Also let the inclination of OU with OX be θ.

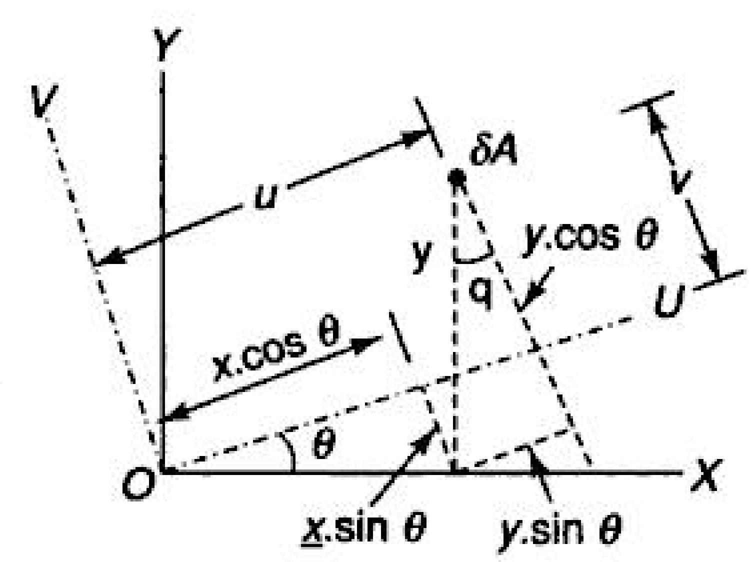


Fig. 5.40

Let δA be an elemental area and

x and y = coordinate of the area relative to OX, OY

u and v = coordinate of the area relative to OU, OV

Then $\quad u = x\cos\theta + y\sin\theta \quad$ And $\quad v = y\cos\theta - x\sin\theta$

Product of inertia = I_{uv}

$$= \int uv dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) dA$$

$$= \int (xy\cos^2\theta - x^2\sin\theta\cos\theta + y^2\sin\theta\cos\theta - xy\sin^2\theta) dA$$

$$= \sin\theta\cos\theta\left(\int y^2 dA - \int x^2 dA\right) + \left(\cos^2\theta - \sin^2\theta\right)\int xy dA$$

$$= \sin\theta\cos\theta\left(\int y^2 dA - \int x^2 dA\right) + \left(\frac{1+\cos\theta}{2} - \frac{1-\cos\theta}{2}\right)\int xy dA$$

$$= \left(\frac{1}{2}\sin 2\theta\right)\left(I_x - I_y\right) + \cos 2\theta . I_{xy} \qquad (5.19)$$

Applying the condition for principal axes, i.e. $I_{uv} = 0$

or $$\left(\frac{1}{2}\sin 2\theta\right)\left(I_x - I_y\right) + \cos 2\theta . I_{xy} = 0$$

or $$\sin 2\theta\left(I_x - I_y\right) = -2\cos 2\theta . I_{xy} \qquad (i)$$

For C, $u = OD = 90.44$ mm; $v = CD = AD = 212.69$ mm; v negative, u positive.

$$\sigma_c = -32.253 + 19.382 = -12.871 \text{ MPa (compressive)}$$

Inclination of the neutral axis, $\tan^{-1}\alpha = \dfrac{v}{u} = -\dfrac{0.0914}{0.3566} = -0.2563$ or $\alpha = -14.38°$

Neutral axis has been shown in the figure, on the upper side of which are compressive stresses and on lower side are tensile stresses.

Example 5.32 *A 60 mm × 40 mm × 6 mm angle is used as a cantilever with a 40-mm leg horizontal and on the top. The length of the cantilever is 600 mm. Determine the position of the neutral axis and the maximum stress developed if a load of 1 kN is applied at the free end. Assume the centre line of load to pass through the shear centre.*

Solution As the centre line of the load passes through the shear centre, no twisting moment acts (Refer section 6.5). The load acts as shown in Fig. 5.43.

To locate the centroid, take moments about the left and upper edge,

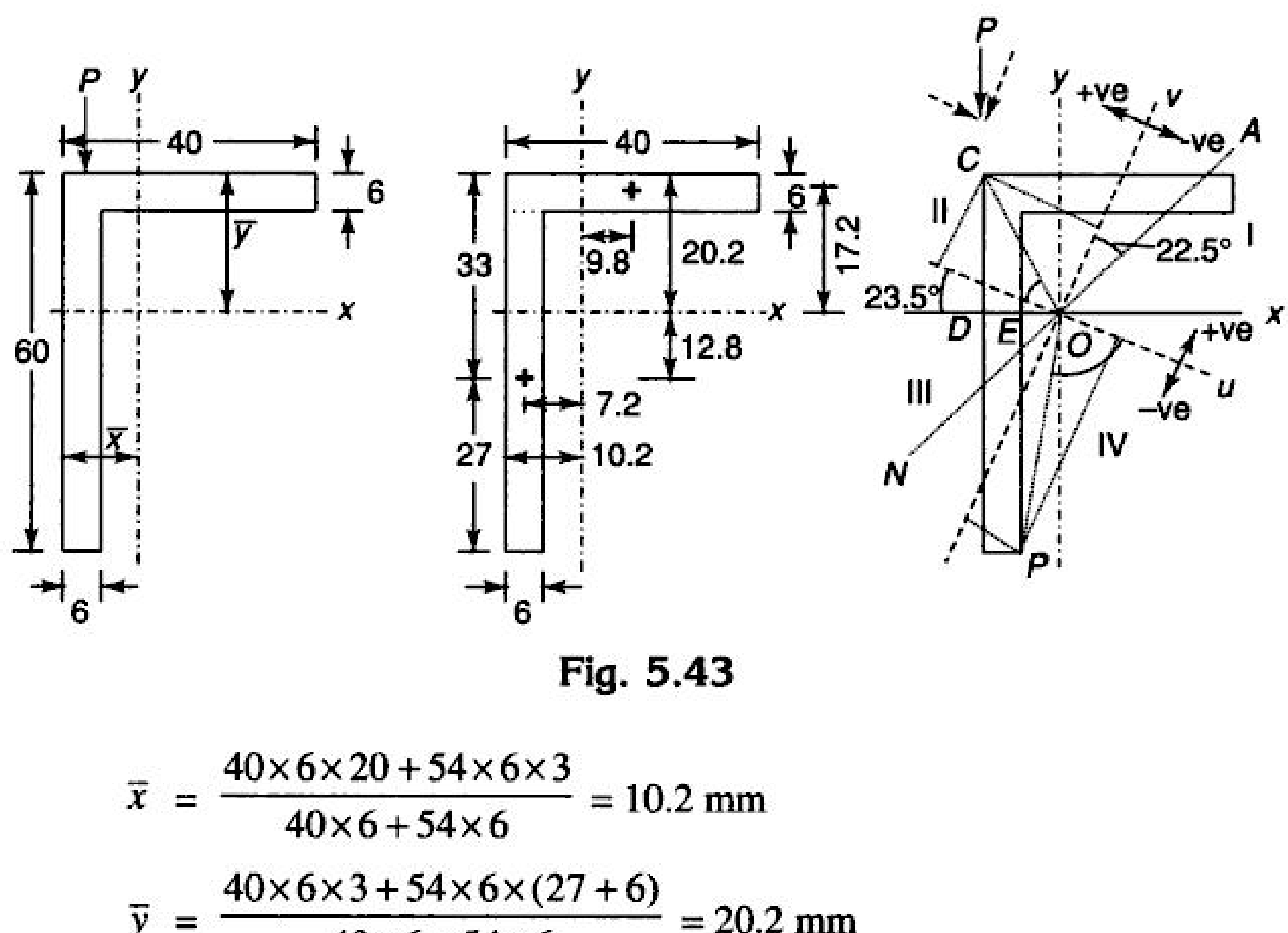


Fig. 5.43

$$\bar{x} = \frac{40\times6\times20 + 54\times6\times3}{40\times6 + 54\times6} = 10.2 \text{ mm}$$

$$\bar{y} = \frac{40\times6\times3 + 54\times6\times(27+6)}{40\times6 + 54\times6} = 20.2 \text{ mm}$$

$$I_x = \frac{6\times54^3}{12} + 6\times54\times12.8^2 + \frac{40\times6^3}{12} + 40\times6\times17.2^2 = 203\,537 \text{ mm}^4$$

$$I_y = \frac{54\times6^3}{12} + 54\times6\times7.2^2 + \frac{6\times40^3}{12} + 6\times40\times9.8^2 = 72\,818 \text{ mm}^4$$

$$I_{xy} = 54\times6\times(-7.2)(-12.8) + 40\times6\times9.8\times17.2 = 70\,314 \text{ mm}^4 \text{...(Eq. 5.24)}$$

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2\times70\,314}{72\,818 - 203\,537} = -1.0758 \text{ or } 2\theta = -47° \text{ or } \theta = -23.5°$$

$$\angle AOE = \cos^{-1}\frac{OE}{OA} = \cos^{-1}\frac{14.56}{96.544} = \cos^{-1}0.1508 = 81.33°$$

$$\angle AOV' = 81.33° - (90° - 26.82°) = 18.15°$$

Thus $v = 96.544 \times \cos 18.15° = 91.74$ mm

and $u = 96.544 \times \sin 18.15° = 30.07$ mm

$$\sigma = 1.066 \times 91.74 + 1.129 \times 30.07 = 131.74 \text{ MPa}$$

5.9 ELLIPSE OF INERTIA OR MOMENTAL ELLIPSE

This is a graphical method which can be used to find moments of inertia about two mutually perpendicular axes through the centroid when moments of inertia about the principal axes are known. The method is as follows:

From equations 5.19, 5.21 and 5.22,

$$I_{uv} = \left(\frac{1}{2}\sin 2\theta\right)\left(I_x - I_y\right) + \cos 2\theta . I_{xy}$$

$$I_u = \cos^2\theta . I_x + \sin^2\theta . I_y - \sin 2\theta . I_{xy}$$

$$I_v = \cos^2\theta . I_y + \sin^2\theta . I_x + \sin 2\theta . I_{xy}$$

If XX and YY are the principal axes, I_{xy} is zero, and UU, VV are any other mutually perpendicular axes. The above equations are changed to

$$I_{uv} = \left(\frac{1}{2}\sin 2\theta\right)\left(I_x - I_y\right)$$

$$I_u = \cos^2\theta . I_x + \sin^2\theta . I_y$$

$$I_v = \cos^2\theta . I_y + \sin^2\theta . I_x$$

The first two equations are similar to equations for shear stress and direct stress at an angle when a material is subjected to two perpendicular stresses (refer section 2.6).

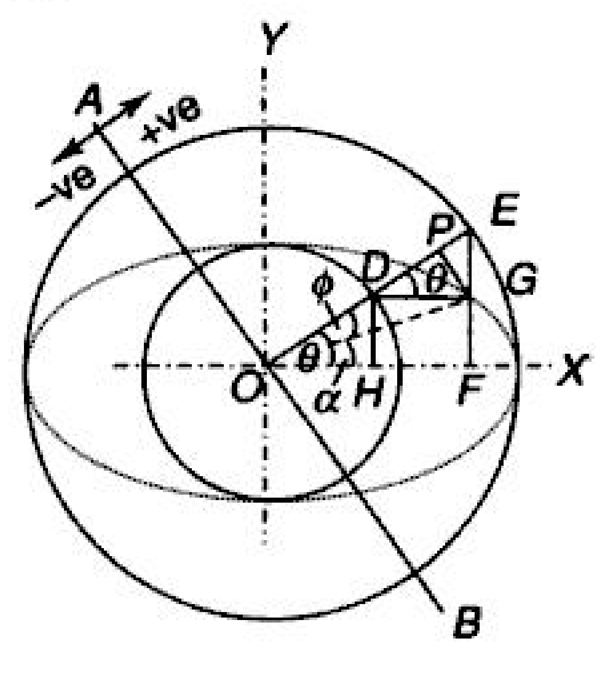


Fig. 5.47

Draw two circles with O as centre and radii equal to I_x and I_y taken to a suitable scale (Fig. 5.47) and complete the diagram.

Now, $OP = OD + DP = OD + DG\cos\theta$

$= I_y + (DE\cos\theta)\cos\theta = I_y + (\sigma_x - \sigma_y)\cos^2\theta$

$= I_x\cos^2\theta + I_y(1 - \cos^2\theta) = I_x\cos^2\theta + I_y\sin^2\theta$

$= I_u$ (Refer Eq. 2.5)

and $PG = DG\sin\theta = (DE\cos\theta)\sin\theta$

$= (I_x - I_y)\cos\theta\sin\theta = \frac{1}{2}(I_x - I_y)\sin 2\theta = I_{uv}$

The diagram is known as ellipse of inertia.

I_v can be found from the relation, $I_v = I_x + I_y - I_u$

- The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the *moment of resistance.*
- Moment of inertia of a rigid body is obtained by summing the products of its various particles with the square of their distances from a given axis.
- *Parallel axis theorem* states that the moment of inertia about any axis parallel to the centroidal axis is equal to the moment of inertia through the centroidal axis plus the product of the area of the figure and the square of the distance between the two axes.
- Moment of inertia and section modulus of different sections are:

Rectangle:	$I_{xx} = bd^3/12$, $I_{ab} = bd^3/3$ and $Z_x = bd^2/6$
Hollow rectangle:	$I_{xx} = (BD^3 - bd^3)/12$, $Z_x = (BD^3 - bd^3)/6D$
I-section:	$I_{xx} = (BD^3 - bd^3)/12$, $Z_x = (BD^3 - bd^3)/6D$
Triangular section:	$I_{xx} = bd^3/36$; $I_{ab} = bd^3/12$
Circular section:	$I_{xx} = \pi d^4/64$; $Z_x = \pi d^3/32$
Hollow circular section:	$I_{xx} = \pi(D^4 - d^4)/64$; $Z_x = \pi(D^4 - d^4)/32D$

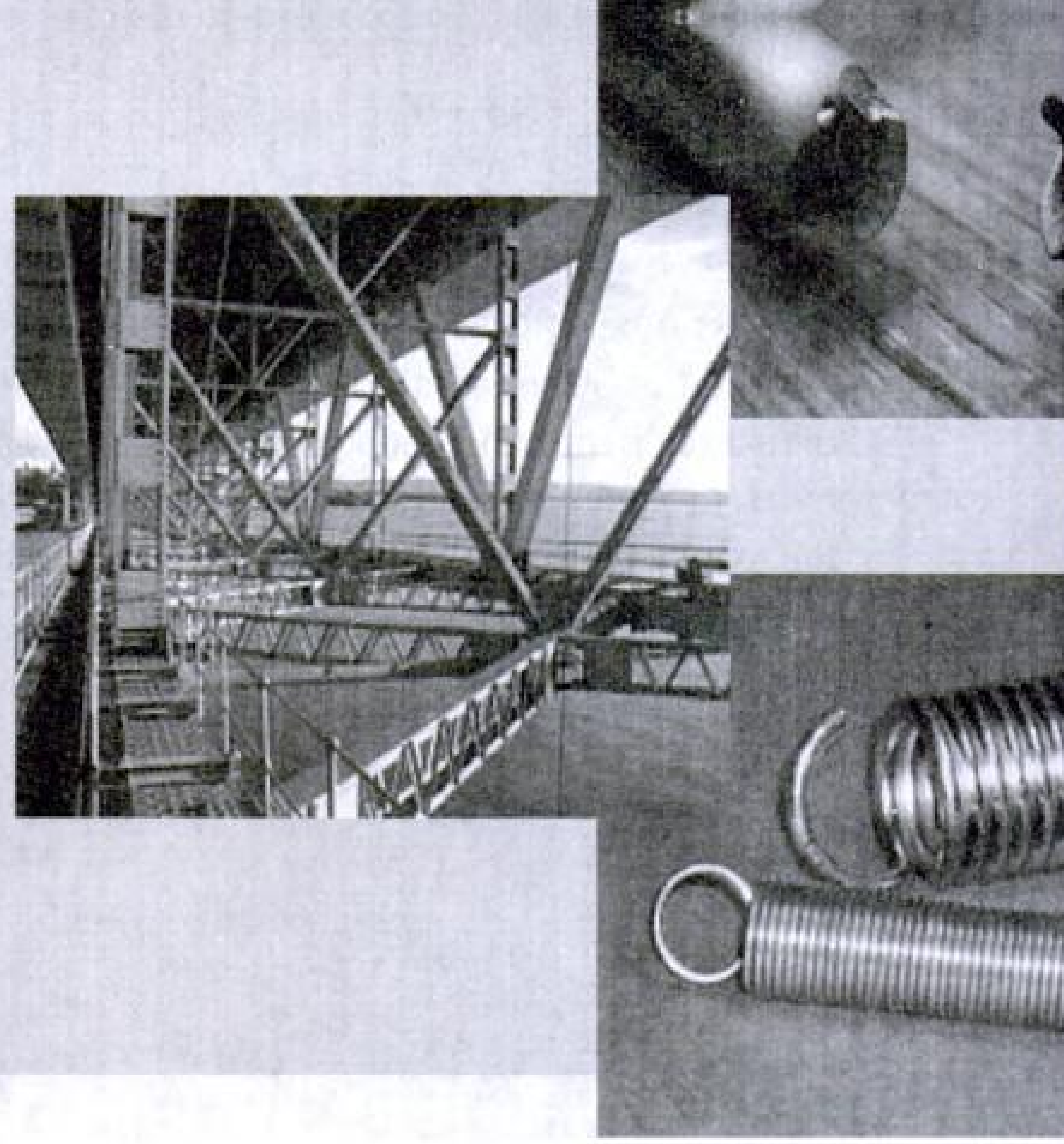
- Beams made up of two different materials such as wooden beams reinforced by steel plates are known as *flitched* or *composite beams.*
- At any common surface in a flitched beam, strain = $\sigma_1/E_1 = \sigma_2/E_2$
- Moment of resistance of a flitched beam, $M_r = \sigma_{1m}(I_1 + mI_2)/y_1$ where m = modular ratio E_2/E_1 and $I_1 + mI_2$ is known as *equivalent moment of inertia* of the cross-section
- To compensate for the weakness of concrete, steel reinforcement is done on the tension side of concrete beams and to have the maximum advantage it is put at the greatest distance from the neutral axis of the beam.
- The integral $\int xy\, dA = 0$ is known as *product of inertia* and the axes for which it is zero for a section are known as *principal axes* of the cross-section.
- The limitation of the theory of bending moment is that it can be applied only to the case of bending about a principal axis.
- The directions of the principal axes can be found from the relation $\tan 2\theta = 2I_{xy}/(I_y - I_x)$; where I_{xy} for a rectangle with sides parallel to the principal axes is given by, $I_{xy} = A.hk$

  Then $I_u = \frac{1}{2}[(I_x + I_y) + \sec 2\theta(I_x - I_y)]$

  and $I_v = \frac{1}{2}[(I_x + I_y) - \sec 2\theta(I_x - I_y)]$
- *Ellipse of inertia* or *momental ellipse* is a graphical method which can be used to find moments of inertia about two mutually perpendicular axes through the centroid when moments of inertia about the principal axes are known.
- In masonry columns, it is desirable that no tensile stresses are set up. It is ensured if the line of action of the load lies within a central area of the section. In rectangular sections, it should lie within the middle third and in circular sections within middle quarter or within a circle of diameter $d/4$ with centre O.

6

SHEAR STRESS IN BEAMS



6.1 INTRODUCTION

While discussing the theory of simple bending in the previous chapter, it was assumed that no shear force acts on the section. However, when a beam is loaded, the shear force at a section is always present along with the bending moment. It is, therefore, important to study the variation of shear stress in a beam and to know its maximum value within safe limits. It is observed that in most cases, the effect of shear stress is quite small as compared to the effect of bending stress and it may be ignored. In some cases, however, it may be desirable to consider its effect also. Usually, beams are designed for bending stresses and checked for shear stresses. This chapter discusses the shear stress and its variation across the section.

A shear force in a beam at any cross-section sets up shear stress on transverse sections the magnitude of which varies across the section. In the analysis, it is assumed that the shear stress is uniform across the width and does not affect the distribution of bending stress. The latter assumption is not strictly true as the shear stress causes a distortion of transverse planes and they do not remain plane.

As every shear stress is accompanied by an equal complimentary shear stress, shear stress on transverse planes has complimentary shear stress on longitudinal or horizontal planes parallel to the neutral axis.

6.2 VARIATION OF SHEAR STRESS

Figure 6.1 shows two transverse sections of a beam at a distance δx apart. Considering the complimentary shear stress τ at a distance y_o from the neutral axis, let F, $F + \delta F$ and M, $M + \delta M$ be the shear forces and the bending moments at the two sections. z is the width of the cross-section at this position.

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Thin Circular Tube

If the thickness of a circular tube is small, then the fact that the shear stress follows the direction of boundary can be used to find the same.

Let the bending be about XX and A and B two symmetrically placed positions at angle θ from vertical (Fig. 6.7). Let the shear stress at A and B be τ. Now, the complimentary shear stress is on longitudinal planes and is balanced by difference of normal stresses on the area subtended by angle 2θ.

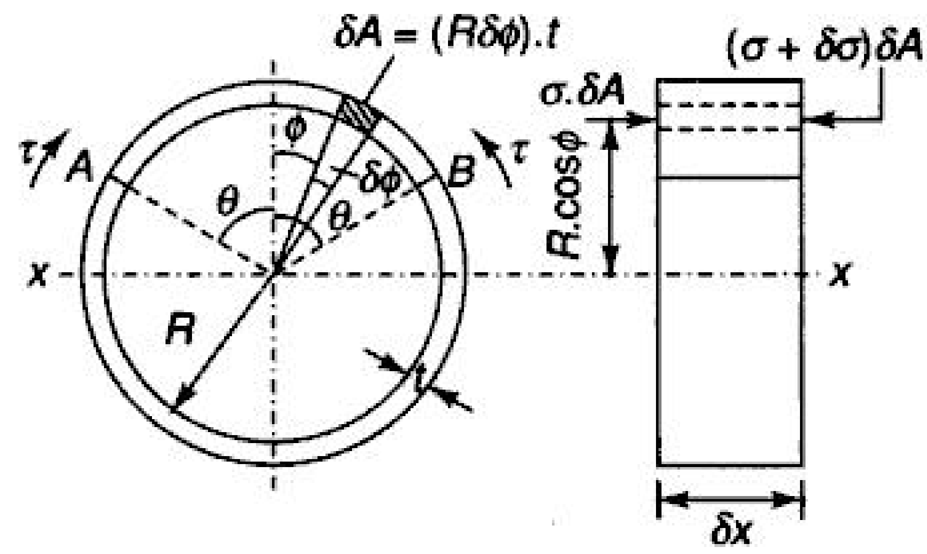

Fig. 6.7

The force due to complimentary shear stress on the area at A and B tends to slide the block above which is resisted by the difference of the longitudinal forces over the area above AB.

Thus,

For a length δx of the beam,

$$2\tau.t.\delta x = \int_{-\theta}^{\theta} d\sigma.dA = \int_{-\theta}^{\theta} d\sigma.(R.d\varphi).t$$

or $$\tau = \frac{1}{2}\int_{-\theta}^{\theta} \frac{\delta\sigma}{\delta x}.(R.d\varphi)$$

But $$\delta\sigma = \frac{\delta M.y}{I}$$

$\therefore$ $$\tau = \frac{1}{2}\int_{-\theta}^{\theta} \frac{\delta M}{\delta x}.(R.d\varphi).\frac{y}{I} = \frac{1}{2}\int_{-\theta}^{\theta} F.(R.d\varphi).\frac{y}{I} \quad .... \quad \left(F = \frac{\delta M}{\delta x}\right)$$

$$I_x = \frac{1}{2}.\text{Polar moment of inertia}$$

$$= \frac{1}{2}\times \text{Area}\times(\text{mean radius})^2 = \frac{1}{2}.2\pi Rt.R^2 = \pi tR^3$$

Hence

$$\tau = \frac{FR}{2I}\int_{-\theta}^{\theta} y.d\varphi = \frac{FR}{2\pi tR^3}\int_{-\theta}^{\theta} R\cos\varphi.d\varphi$$

$$= \frac{F}{2\pi Rt}\int_{-\theta}^{\theta}\cos\varphi.d\varphi = \frac{F}{2\pi Rt}(\sin\theta)_{-\theta}^{\theta} = \frac{F\sin\theta}{\pi Rt}$$

or $\tau = 2X$ mean shear stress (6.9)

Square with a Diagonal Horizontal

Refer Fig. 6.8,

$$I_{xx} = 2\left[\frac{B.(B/2)^3}{12}\right] = \frac{B^4}{48}$$

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Example 6.2 *A simply supported beam of 2-m span carries a uniformly distributed load of 140 kN per m over the whole span. The cross-section of the beam is a T-section with a flange width of 120 mm, web and flange thickness 20 mm and overall depth 160 mm. Determine the maximum shear stress in the beam and draw the shear stress distribution for the section.*

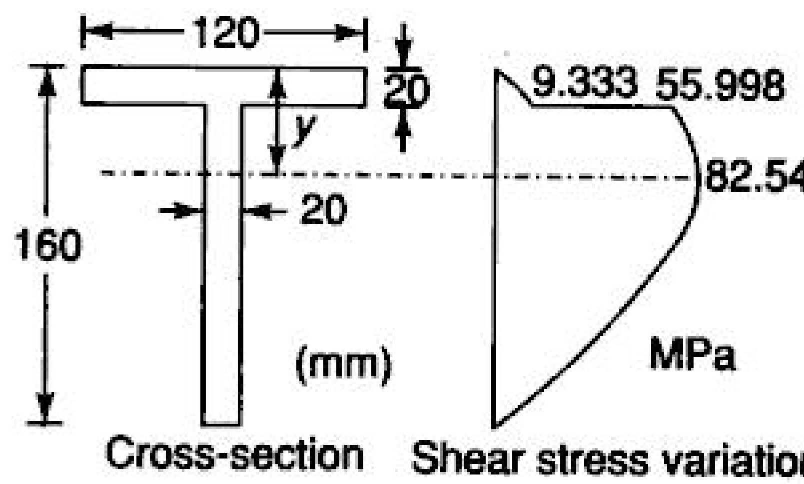


Fig. 6.12

Solution Refer Fig. 6.12, $F_{max} = 140 \times 1 = 140$ kN
Taking moments about the top edge,

$$\bar{y} = \frac{120\times20\times10+140\times20\times90}{120\times20+140\times20} = 53.08 \text{ mm}$$

$$I = \frac{120(20)^3}{12} + 120\times20\times(53.08-10)^2 + \frac{20(140)^3}{12} + 20\times140(90-53.08)^2$$

$$= 12.924 \times 10^6 \text{ mm}^4$$

Shear stress in the flange at the junction

$$= F.\frac{A\bar{y}}{zI} = \frac{140\times10^3\times(120\times20)(53.08-10)}{12.924\times10^6\times120} = 9.333 \text{ MPa}$$

Shear stress in the web at the junction $= 9.333\times\dfrac{120}{20} = 56$ MPa

Maximum shear stress (at N.A.)

$$= \frac{140\times10^3\times[120\times20\times43.08+(20\times70)\times35]}{12.924\times10^6\times20} = 82.54 \text{ MPa}$$

Example 6.3 *Figure 6.13 shows a cast iron bracket subjected to bending. If the maximum tensile stress in the top flange is not to exceed 15 MPa, determine the bending moment the section can take. If the beam is subjected to a shear force of 150 kN, sketch the stress distribution over the depth of the section.*

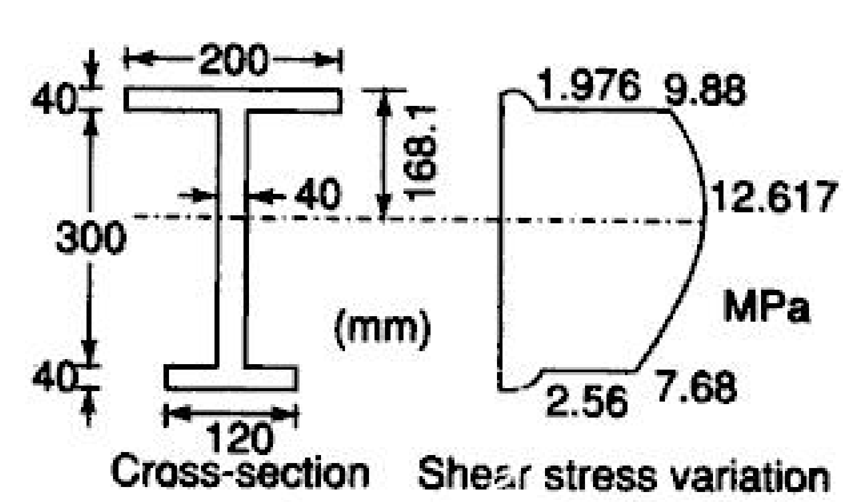


Fig. 6.13

Solution Taking moments about the top edge,

$$\bar{y} = \frac{200\times40\times20+300\times40\times190+120\times40\times360}{200\times40+300\times40+120\times40} = 168.1 \text{ mm}$$

$$I = \frac{200\times40^3}{12} + 200\times40(168.1-20)^2 + \frac{40\times300^3}{12} + 40\times300(190-168.1)^2$$

$$+\frac{120\times40^3}{12} + 120\times40(360-168.1)^2$$

(iv) At the neutral axis, $\sigma = 0$

$$\tau = \frac{25\times10^3\times(160\times150\times75)}{360\times10^6\times160} = 0.781 \text{ MPa}$$

Principal stresses, $\sigma_1 = \sigma_2 = 0.781$ MPa; $\theta_1 = \theta_2 = 45°$

At the neutral axis, there is a state of simple shear. Principal stresses are 0.76 MPa compressive along one diagonal plane and 0.76 MPa tensile along another diagonal plane.

Example 6.7 *A 320 mm × 160 mm I-section joist has 20-mm thick flanges and a 15-mm thick web. At a certain cross-section it is acted upon by a bending moment of 100 kN.m and a shear force of 200 kN. Determine the principal stresses*

(i) at the top
(ii) in the flanges at 140 mm from neutral axis
(iii) in the web at 140 mm from neutral axis
(iv) at the neutral axis

Plot the variations along the section.

Solution Refer Fig. 6.17,

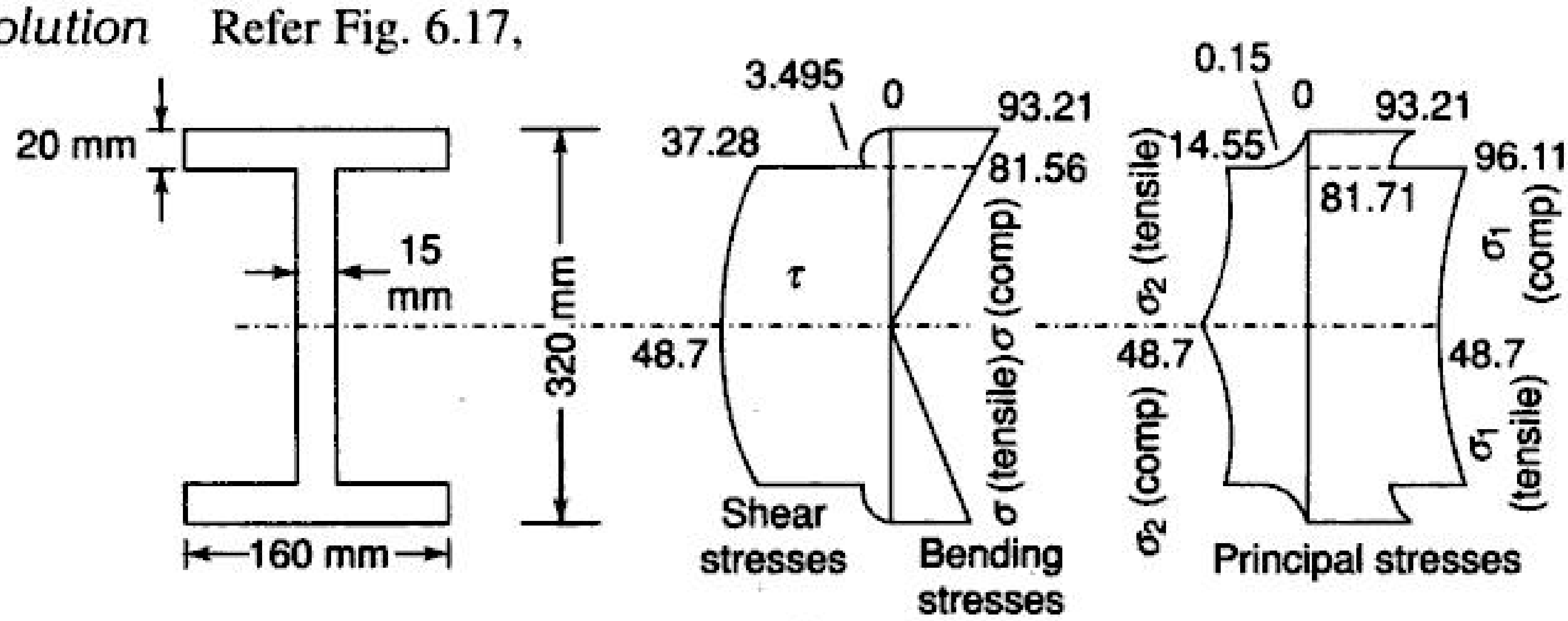


Fig. 6.17

$$I = \frac{160\times320^3}{12} - \frac{145\times280^3}{12} = 171.65\times10^6 \text{ mm}^4$$

(i) At the top, $\sigma = \dfrac{100\times10^6\times160}{171.65\times10^6} = 93.21$ MPa; $\tau = 0$,

Principal stresses, $\sigma_1 = 93.21$ MPa; $\sigma_2 = 0$

(ii) At 140 mm from neutral axis, $\sigma = 93.21\times\dfrac{140}{160} = 81.56$ MPa

$$\tau = \frac{F}{8I}(D^2 - d^2) \qquad \text{...(Eq. 6.3)}$$

$$= \frac{200\times10^3}{8\times171.65\times10^6}(320^2 - 280^2) = 3.495 \text{ MPa}$$

Principal stresses, $\sigma_{1,2} = \dfrac{81.56}{2} \pm \sqrt{\left(\dfrac{81.56}{2}\right)^2 + 3.495^2} = 40.78 \pm 40.93$

$= 81.71$ MPa and -0.15 MPa

Horizontal shear stress at level 2,

$$\tau = \frac{F.A\bar{y}}{Ib} = \frac{5000 \times 120 \times 40 \times 80}{80 \times 10^6 \times 120} = 0.2 \text{ MPa}$$

Load carried by each bolt = $\tau \times$ area = $\tau \times b \times$ spacing

$$\text{Shear stress in the bolt} = \frac{\tau \times b \times \text{spacing}}{\text{Area of bolt}} = \frac{0.2 \times 120 \times 110}{314.2} = 8.402 \text{ MPa}$$

$$\text{Horizontal shear stress at level 3, } \tau = \frac{5000 \times 120 \times 80 \times 60}{80 \times 10^6 \times 120} = 0.3 \text{ MPa}$$

$$\text{Shear stress in the bolt} = \frac{0.3 \times 120 \times 110}{314.2} = 12.603 \text{ MPa}$$

Example 6.10 *A composite beam consists of a 180 mm × 140 mm timber section bonded with 10 mm × 140 mm steel plates at top and bottom. Determine the stresses in the beam when it is subjected to a shear force of 100 kN. Also find the spacing of bolts of 12 mm diameter for the shear connection between the flitches and the timber beam. Allowable shear stress in steel is 100 MPa. The Young's modulus of steel is 210 GPa and of timber 15 GPa.*

Solution Refer Fig. 6.22,

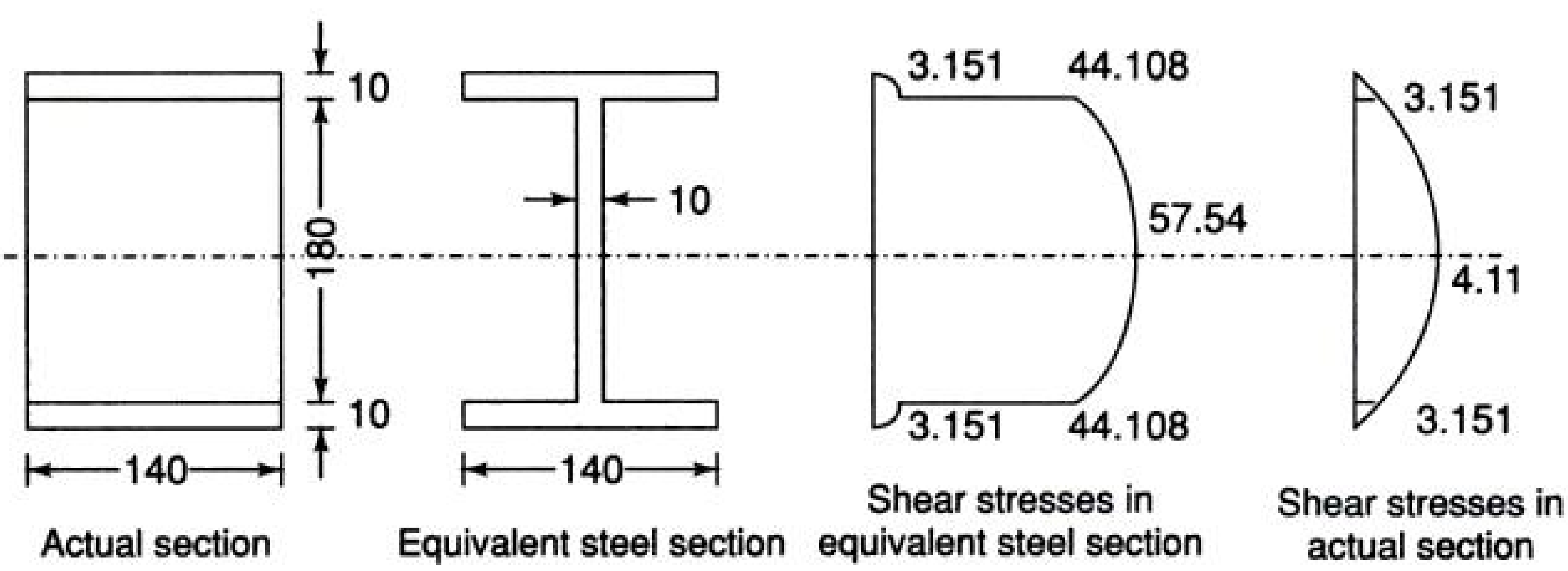


Fig. 6.22

$$\text{Modular ratio} = \frac{E_s}{E_t} = \frac{210}{15} = 14$$

Transforming the composite beam into an equivalent steel section,

$$I_s = \frac{10 \times 180^3}{12} + 2\left[\frac{140 \times 10^3}{12} + 140 \times 10 \times 95^2\right] = 30.153 \times 10^6 \text{ mm}^4$$

Shear stress in the equivalent steel section,
At steel plate timber junction (in the flanges)

$$\tau = F.\frac{A\bar{y}}{Iz} = 100 \times 10^3 \times \frac{140 \times 10 \times 95}{30.153 \times 10^6 \times 140} = 3.151 \text{ MPa}$$

At steel plate timber junction (in the web) = 3.151 × 14 = 44.108 MPa

Solution

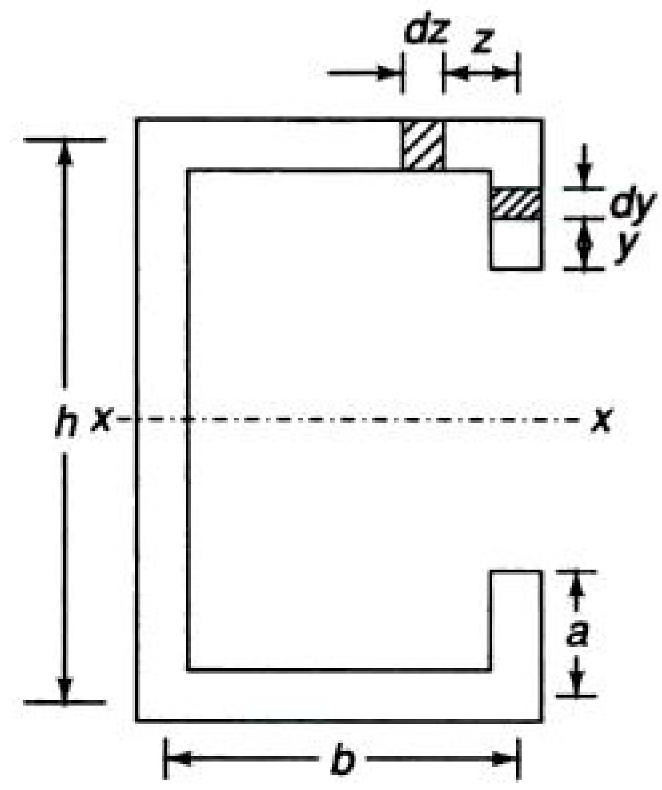


Fig. 6.26

$$F_1 = \int \tau.dA = \int \frac{F}{tI} A\bar{y}.(t.dy)$$

$$= \frac{F}{tI}\int (ty)\left(\frac{h}{2} - a + \frac{y}{2}\right).tdy$$

$$= \frac{Ft}{2I}\int_0^a \left(hy - 2ay + y^2\right)dy$$

$$= \frac{Ft}{2I}\left[h\frac{y^2}{2} - 2a\frac{y^2}{2} + \frac{y^3}{3}\right]_0^a$$

$$= \frac{Fta^2}{12I}(3h - 4a)$$

$$F_2 = \int \tau.dA = \int \frac{F}{tI} A\bar{y}.(t.dz) = \frac{F}{tI}\int\left[at\left(\frac{h}{2} - \frac{a}{2}\right) + tz.\frac{h}{2}\right].tdz$$

$$= \frac{Ft}{2I}\int_0^b a\left(h - a\right) + hz\right)dz = \frac{Ft}{2I}\left[a(h-a).z + h\frac{z^2}{2}\right]_0^b = \frac{Ftb}{4I}[2a(h-a) + hb]$$

Taking moments about point D,

$$F.e = 2F_1.b + 2F_2.\frac{h}{2} = 2F_1.b + F_2.h = 2\frac{Fta^2}{12I}(3h - 4a).b + \frac{Ftb}{4I}[2a(h-a) + hb]h$$

$$e = \frac{bt}{12I}\left(6ha^2 - 8a^3 + 6ah^2 - 6ha^2 + 3h^2b\right)$$

$$= \frac{bt}{12I}\left(6ah^2 - 8a^3 + 3h^2b\right)$$

$$I_x = 2\left[\frac{ta^3}{12} + ta\left(\frac{h}{2} - \frac{a}{2}\right)^2\right] + 2\left[\frac{bt^3}{12} + bt\left(\frac{h}{2}\right)^2\right] + \frac{th^3}{12}$$

$$= t\left[\frac{a^3}{6} + \frac{a}{2}\left(h^2 - 2ah + a^2\right) + \frac{bt^2}{6} + \frac{bh^2}{2} + \frac{h^2}{12}\right]$$

$$= \frac{t}{12}\left[8a^3 + 6ah^2 - 12a^2h + 2bt^2 + 6bh^2 + h^3\right]$$

$$e = \frac{b(6ah^2 - 8a^3 + 3h^2b)}{(8a^3 + 6ah^2 - 12a^2h + 2bt^2 + 6bh^2 + h^3)}$$

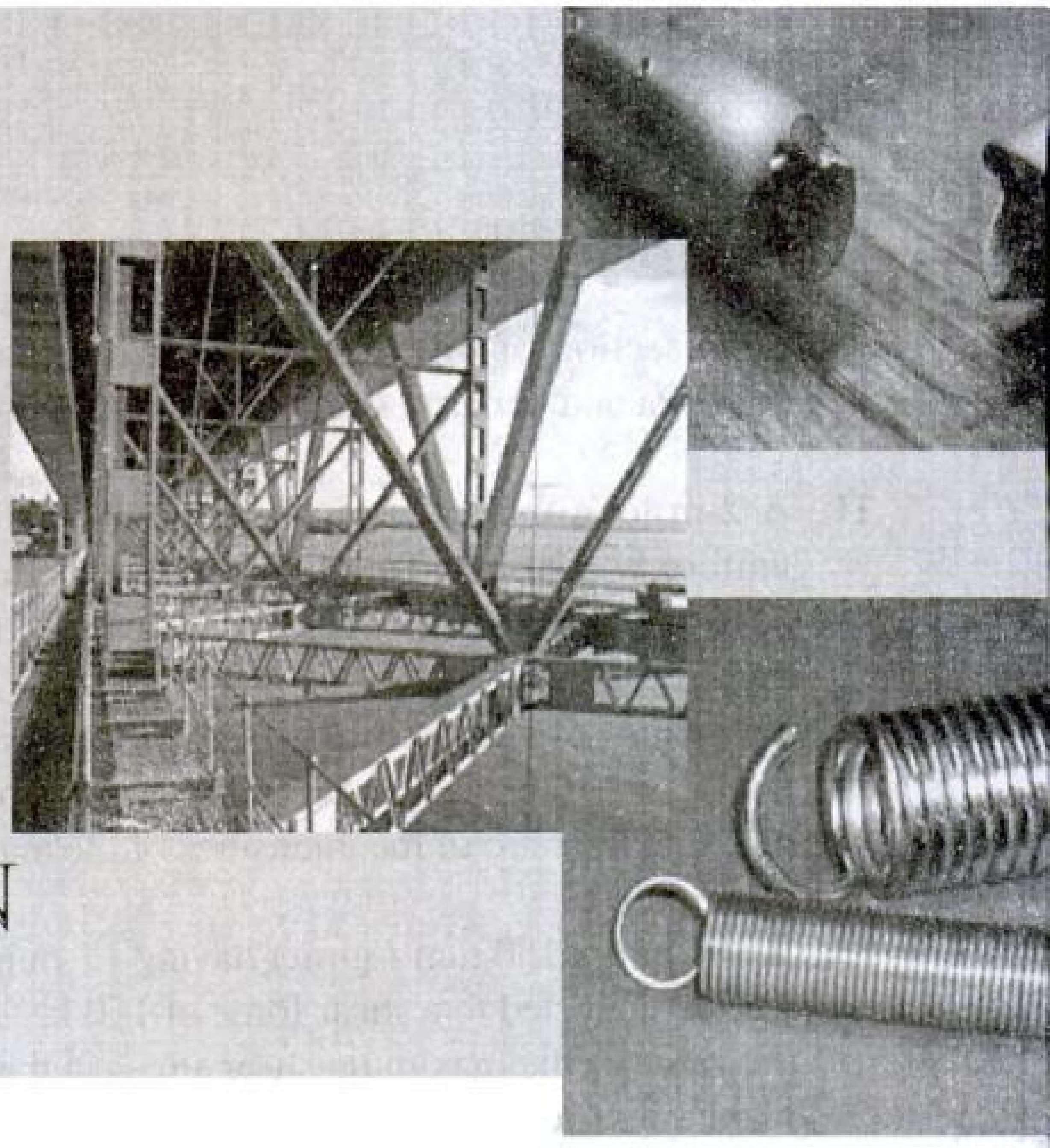
If $a = 0$, $$e = \frac{b(3h^2b)}{(2bt^2 + 6bh^2 + h^3)}$$

And neglecting moment of inertia of flanges about their axes,

$$e = \frac{3b^2}{6b + h}$$ as obtained earlier.

7

SLOPE AND DEFLECTION



7.1 INTRODUCTION

As a load is applied on a beam, it deflects. The deflection can be observed and measured directly whereas other parameters such as shear force, bending moment and stresses can only be calculated. Though it is important that the cross-section of a beam is strong enough to withstand the bending stresses and shear stresses, i.e. it is based on *strength criterion*, the deflections must also be restricted. Excessive deflections can cause visible or invisible cracks in beams. Also, excessive deflections perceptible by naked eye give a feeling of unsafe structure to the occupants of the building causing adverse effect on their health. Thus it is extremely important to have the knowledge of maximum deflection in a beam under the given loading. The maximum deflection of a beam must not exceed a given limit. The designing of a beam from this aspect is known as *stiffness criterion*. In this chapter, the governing differential equation of beams is formulated and various methods of solution are discussed. The basic method involves integrating the differential equation whereas in other methods the integral is obtained indirectly. The relation obtained provides the *elastic curve*, i.e. the curve into which the axis of the beam is transformed under the loading.

7.2 BEAM DIFFERENTIAL EQUATION

As mentioned above, the deflection profile of a beam is known as its *elastic curve*. If a beam is subjected to pure bending, it is bent into a circular arc and the radius of bending or the radius of curvature is given by $\frac{M}{I} = \frac{E}{R}$ or $R = \frac{EI}{M}$. However, the radius of curvature may not be constant at all the points as the beam may not be subjected to pure bending, which is generally the case.

and $y = -\frac{W}{6EI}(3lx^2 - x^3)$ (7.6)

- At the free end, $x = l$, the slope and the deflection are maximum and are given by

Slope $= -\frac{Wl^2}{2EI}$ and Deflection $= -\frac{Wl^3}{3EI}$ (7.6a)

The slope and deflection are shown in Fig. 7.4*b* and *c* respectively.

(ii) Concentrated load not at free end Between *AC*, at any section at a distance *x* from *A* (Fig. 7.5*a*), $M = -W(a-x)$

The equation of slope and elastic curve can be obtained as in previous case in the form

$$y' = -\frac{W}{2EI}(2ax - x^2) \text{ and}$$

$$y = -\frac{W}{6EI}(3ax^2 - x^3), \text{ i.e. by}$$

replacing *l* by *a*.

At *C*, $x = a$, $y' = -\frac{Wa^2}{2EI}$

and $y_c = -\frac{Wa^3}{3EI}$

Between *CB*, at any section at a distance *x* from *A*, $M = 0$,

$$\therefore \; EI\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = 0$$

or $\frac{dy}{dx} = C_1$, i.e. the slope is constant between *CB* and is equal to slope at *C* or the portion of cantilever from *C* to *B* remains straight with slope $\frac{dy}{dx} = y' = \frac{GF}{GE} = -\frac{Wa^2}{2EI}$ or $GF = y'.GE$ (Fig. 7.5*b*).

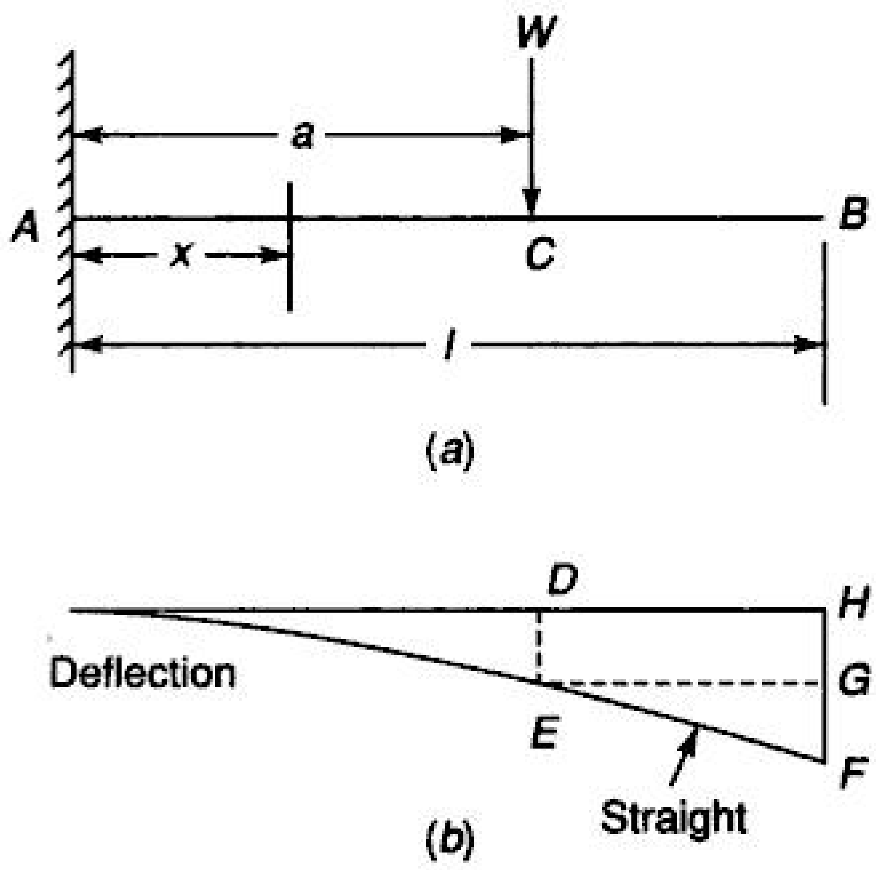


Fig. 7.5

Deflection at *B* = Deflection at *C* + *GF* = Deflection at *C* + $y'.GE$

$$= -\frac{Wa^3}{3EI} - \frac{wa^2}{2EI}.(l-a) \quad (7.7)$$

If *W* is at the midpoint, deflection $= \left[\frac{W(l/2)^3}{3EI} + \frac{W(l/2)^2}{2EI}.\frac{l}{2}\right] = \frac{5Wl^3}{48EI}$ (7.7a)

(iii) Uniformly distributed load on whole span Let origin be at the free end (Fig. 7.6*a*). At a section at a distance *x* from the free end,

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

- Integrating, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$

(a) For whole span having uniformly distributed load, $y_b = -\frac{wl^4}{8EI}$ (Refer Eq. 7.10)

(b) For span loaded with varying intensity, $y_b = -\frac{wl^4}{30EI}$ (Refer Eq. 7.14)

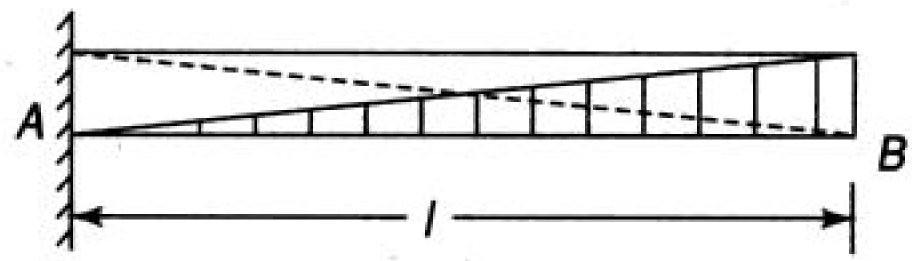


Fig. 7.11

Thus deflection of $B = -\frac{wl^4}{8EI} - \left(-\frac{wl^4}{30EI}\right) = -\frac{11wl^4}{120EI}$ (7.15)

Simply Supported Beam

(i) Concentrated load at midspan

Figure 7.12*a* shows a simply supported beam *AB* of span *l* carrying a load *W* at the midpoint *C*.

$\therefore \quad R_a = R_b = W/2$

Consider a section of the cantilever from *A* (origin at *A*),

$$M = \frac{W}{2}x \quad \text{(Positive, being sagging)}$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = \frac{W}{2}x$$

- Integrating, $EI\frac{dy}{dx} = \frac{Wx^2}{4} + C_1$

At $x = \frac{l}{2}, \frac{dy}{dx} = 0,$

$$\therefore \ C_1 = -\frac{Wl^2}{16} \quad \therefore \ EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}$$

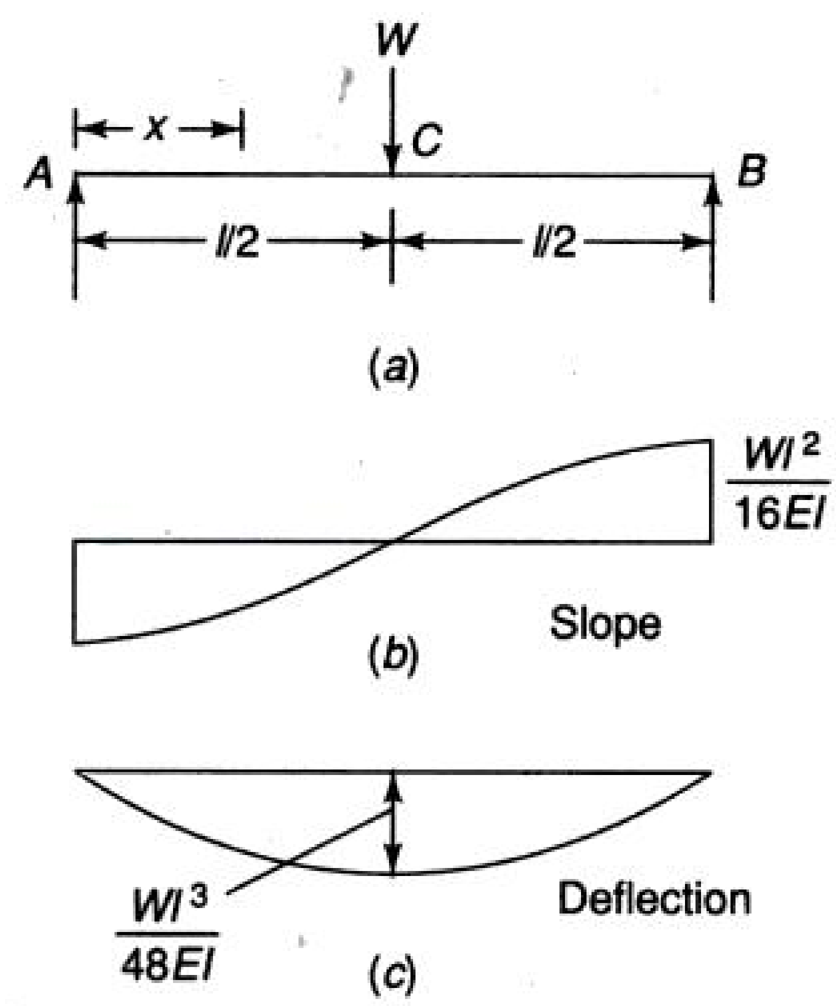


Fig. 7.12

- Integrating again, $EIy = \frac{Wx^3}{12} - \frac{Wl^2x}{16} + C_2$

At $x = 0, y = 0, \quad \therefore \ C_2 = 0 \quad \therefore \ EIy = \frac{Wx^3}{12} - \frac{Wl^2x}{16}$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = -\frac{W}{16EI}(l^2 - 4x^2) \text{ and } y = -\frac{W}{48EI}(3l^2x - 4x^3)$$

- At *A*, $x = 0, \quad \therefore$ slope $= -\frac{Wl^2}{16EI}$ (7.16)

Deflection at $C = -\frac{W}{48EI}\left(3l^2\frac{l}{2} - 4.\frac{l^3}{8}\right) = -\frac{Wl^3}{48EI}$ (7.17)

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- Integrating again, $EIy = \frac{wlx^3}{36} - \frac{wx^5}{120l} + C_1x + C_2$

At $x = 0, y = 0, \quad \therefore \quad C_2 = 0$

At $x = l, y = 0, \quad \therefore \quad 0 = \frac{wl^4}{36} - \frac{wl^4}{120} + C_1 l \quad$ or $\quad C_1 = -\frac{7wl^3}{360}$

Thus $EIy = \frac{wlx^3}{36} - \frac{wx^5}{120l} - \frac{7wl^3}{360}x$ (7.25)

To find maximum deflection, equate the slope to zero, i.e.

$$EI\frac{dy}{dx} = \frac{wlx^2}{12} - \frac{wx^4}{24l} - \frac{7wl^3}{360} = 0$$

or $30l^2x^2 - 15x^4 - 7l^4 = 0$ or $15x^4 - 30l^2x^2 + 7l^4 = 0$

Let $x = kl$

Then $15k^4 - 30k^2 + 7 = 0$; Solving, $k^2 = \frac{30 \pm \sqrt{900 - 4 \times 7 \times 15}}{30} = 0.2697$

(considering the feasible value of k, it cannot be more than 1)

or $x^2/l^2 = 0.2697$ or $x = 0.5193l$

Thus for maximum deflection,

$$EIy = \frac{wl(0.5193l)^3}{36} - \frac{w(0.5193l)^5}{120l} - \frac{7wl^3}{360}(0.5193l)$$

or $EIy = -0.00652wl^4$ or $y_{max} = -0.00652\frac{wl^4}{EI}$ (7.26)

Example 7.1 *Two parallel steel cantilevers one above the other, each of length l project horizontally from a vertical wall. Their free ends are connected together by a vertical steel tie rod of length a. A load W is applied at the midpoint of the lower beam. Show that the pull in the rod is given by, $P = \frac{5}{32}\cdot\frac{Wl^3}{l^3 + (6al)/(\pi d^2)}$ where d is the diameter of the rod and I the moment of inertia of the section of each beam about its neutral axis.*

If the length of each cantilever is 3 m and that of the tie rod 2.4 m, find the proportion of the load W carried by the tie bar. The diameter of the rod is 20 mm and the moment of inertia of each cantilever is 28×10^6 mm^4.

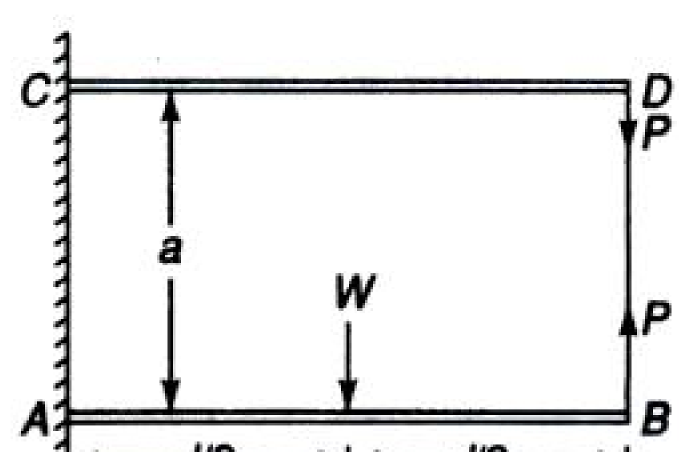

Fig. 7.16

Solution Figure 7.16 shows the cantilevers.

For cantilever CD, $y_1 = \frac{Pl^3}{3EI}$ (downwards) ...(Eq. 7.6a)

For cantilever AB, $y_2 = \frac{5Wl^3}{48EI} - \frac{Pl^3}{3EI}$ (downwards) ...(Eq. 7.7a)

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Example 7.14 *A beam of length l is loaded as shown in Fig. 7.31a. Find the expressions for slope and deflection at any point.*

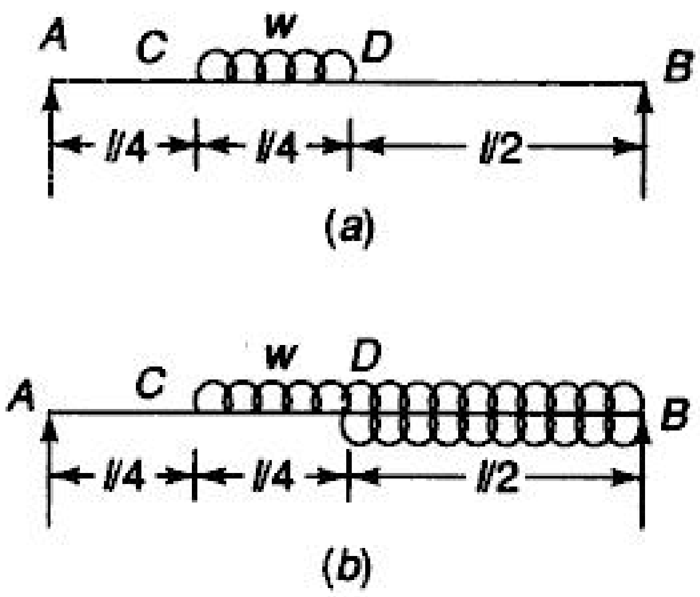

Fig. 7.31

Solution When the load does not reach the end support, it is treated in a way that first it reaches the end and then an upward load is superimposed for that portion where originally there is no load, i.e. the loading is considered downward from C to B and upwards from D to B (Fig.7.31b).

Taking moments about B,

$$R_a.l = w.\frac{l}{4}\left(\frac{l}{2}+\frac{l}{8}\right) \quad \text{or} \quad R_a = \frac{5wl}{32}$$

At any point x from A, $$EI\frac{d^2y}{dx^2} = -\frac{5wl}{32}x \Big| + \frac{w(x-l/4)^2}{2} \Big| - \frac{w(x-l/2)^2}{2}$$

Integrating, $$EI\frac{dy}{dx} = -\frac{5wl}{64}x^2 + C_1 \Big| + \frac{w}{6}\left(x-\frac{l}{4}\right)^3 \Big| - \frac{w}{6}\left(x-\frac{l}{2}\right)^3$$

Integrating again, $$EIy = -\frac{5wl}{192}x^3 + C_1x + C_2 \Big| + \frac{w}{24}\left(x-\frac{l}{4}\right)^4 \Big| - \frac{w}{24}\left(x-\frac{l}{2}\right)^4$$

At $x = 0, y = 0;$ $\therefore$ $C_2 = 0$ (considering first part only)

At $x = l, y = 0;$ $\therefore$ $$0 = -\frac{5wl^4}{192} + C_1l + \frac{w}{24}.\frac{81}{256}l^4 - \frac{w}{24}.\frac{1}{16}l^4 \quad \text{or} \quad C_1 = \frac{95}{6144}wl^3$$

Thus

Slope is $$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{5wl}{64}x^2 + \frac{95}{6144}wl^3 \Big| + \frac{w}{6}\left(x-\frac{l}{4}\right)^3 \Big| - \frac{w}{6}\left(x-\frac{l}{2}\right)^3\right]$$

And deflection is

$$EIy = \frac{1}{EI}\left[-\frac{5wl}{192}x^3 + \frac{95}{6144}wl^3x \Big| + \frac{w}{24}\left(x-\frac{l}{4}\right)^4 \Big| - \frac{w}{24}\left(x-\frac{l}{2}\right)^4\right]$$

Example 7.15 *A simply supported beam has its supports 8 m apart at A and B. It carries a uniformly distributed load of 4 kN/m between A and B starting from 1 m and ending at 5 m from A. The end B of the beam has an overhang of 1 m and at the free end a concentrated load of 8 kN is applied. Determine deflection of the free end and the maximum deflection between A and B. Tale E = 210 GPa and I =* 20×10^6 *mm*^{4}.

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of cantilevers (zero slope at fixed ends), symmetrically loaded simply supported beams (zero slope at the centre) and built-in beams (zero slope at each end).

Examples:

(i) *Cantilever with a concentrated load at the free end*

Figure 7.34 shows the cantilever with the concentrated load W at the free end and its bending moment diagram.

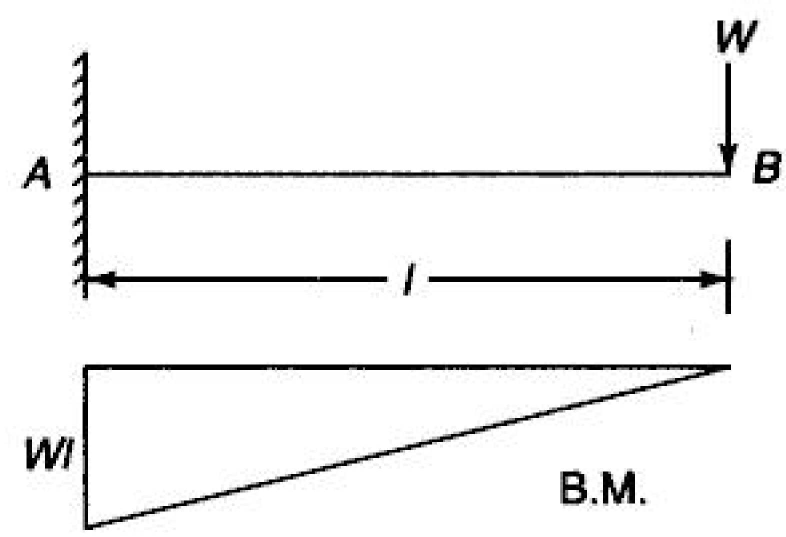

Fig. 7.34

Area of the bending moment diagram,

$$A = \frac{1}{2}l.Wl = \frac{Wl^2}{2}$$

$$\text{Slope at free end} = -\frac{A}{EI} = -\frac{Wl^2}{2EI}$$

$$\text{Deflection} = \frac{A\bar{x}}{EI} = \frac{Wl^2}{2EI}\cdot\frac{2l}{3} = \frac{Wl^3}{3EI} \tag{7.30}$$

(ii) *Cantilever with uniformly distributed load*

Figure 7.35 shows the cantilever with uniformly distributed load and its bending moment diagram.

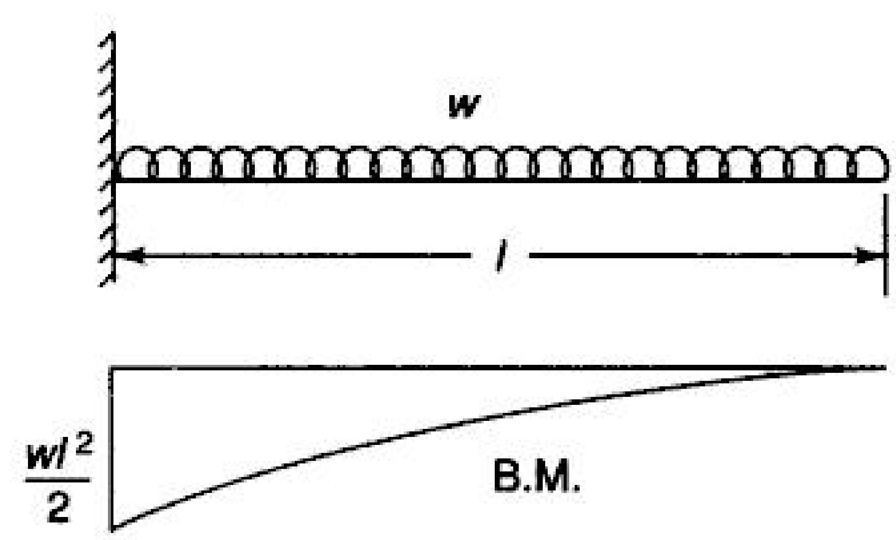

Fig. 7.35

Area of the bending moment diagram, $A = \frac{1}{3}l.\frac{wl^2}{2} = \frac{wl^3}{6}$

$$\text{Slope at free end} = -\frac{A}{EI} = -\frac{wl^3}{6EI}$$

$$\text{Deflection} = \frac{A\bar{x}}{EI} = \frac{wl^3}{6EI}\cdot\frac{3l}{4} = \frac{wl^4}{8EI} \tag{7.31}$$

(iii) *Simply supported beam with concentrated load at the midspan*

As the loading is symmetrical, area of half the bending moment diagram can be considered (Fig. 7.36).

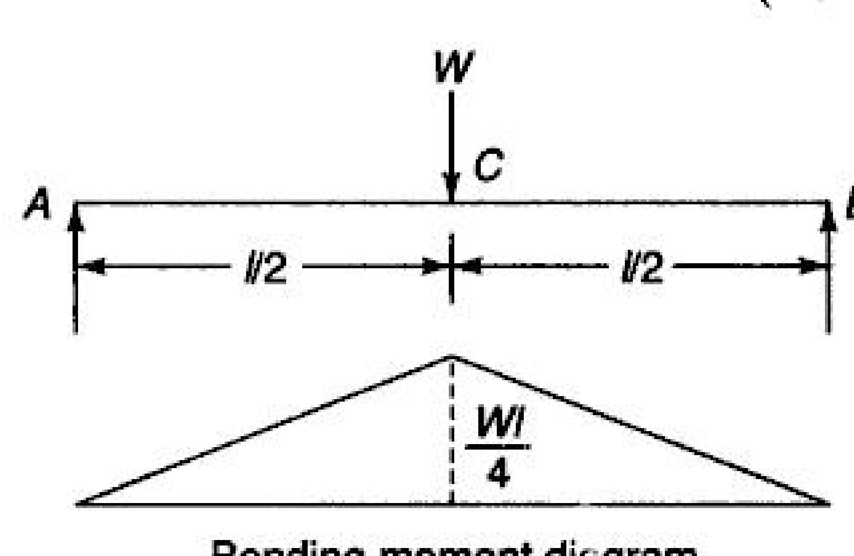

Fig. 7.36

Area of the bending moment diagram,

$$A = \frac{1}{2}\cdot\frac{l}{2}\cdot\frac{Wl}{4} = \frac{Wl^2}{16}$$

$$\text{Slope at free end} = -\frac{A}{EI} = -\frac{Wl^2}{16EI}$$

$\bar{x}$ = distance of centroid of the bending moment diagram from $A = \frac{2}{3}\cdot\frac{l}{2} = \frac{l}{3}$

$$\text{Deflection} = \frac{A\bar{x}}{EI} = \frac{Wl^2}{16EI}\cdot\frac{l}{3} = \frac{Wl^3}{48EI} \tag{7.32}$$

$$= 120.2\times10^{-6}\int_0^{106} x^{2/3}.dx = 120.2\times10^{-6}\left(\frac{x^{5/3}}{5/3}\right)_0^{106}$$

$$= 72.1\times10^{-6}(2374) = 0.171 \text{ mm}$$

7.7 STRAIN ENERGY DUE TO BENDING

Consider two sections of a beam a small distance dx apart (Fig. 7.41). As the distance is small, the bending moment acting may be taken to be same throughout the length dx. Let it be M. Let σ be the bending stress on an elemental cylinder of area dA at a distance y from the neutral axis.

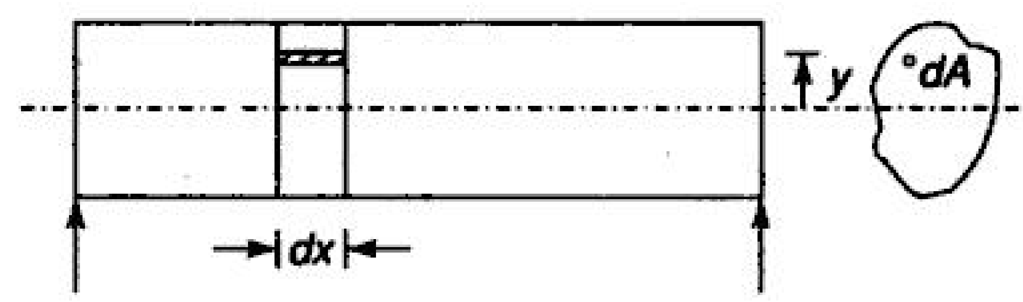

Fig. 7.41

Strain energy in the elemental cylinder = $\frac{\sigma^2}{2E}$ × volume = $\frac{\sigma^2}{2E}\times dA.dx$

Strain energy of the length dx,

$$\delta U = \int\frac{\sigma^2}{2E}.dA.dx = \int\frac{1}{2E}\left(\frac{My}{I}\right)^2.dA.dx = \frac{M^2.dx}{2EI^2}\int y^2.dA$$

But $\int y^2.dA = I$

$$\therefore \quad \delta U = \frac{M^2.dx}{2EI}$$

Strain energy stored in the whole beam, $U = \int\frac{M^2.dx}{2EI}$ (7.34)

7.8 CASTIGLIANO'S FIRST THEOREM (DEFLECTION FROM STRAIN ENERGY)

Castigliano's first theorem is stated as below:

If a structure is subjected to a number of external loads (or couples), the partial derivative of the total strain energy with respect to any load (or couple) provides the deflection in the direction of that load (or couple).

Mathematically,

Let U = total strain energy of the structure

W_1, W_2, W_3 .....External loads at points O_1, O_2, O_3 ....

M_1, M_2, M_3 .....External couples at the same points ....

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Example 7.22 *Compare the strain energy of a centrally loaded simply supported beam with that of the same beam with a uniformly distributed load. Assume the value of the maximum bending stress to be the same in the two cases.*

Solution As maximum bending stress = M/Z, for the same beam in the two cases, maximum M has to be the same, i.e.

$$\frac{Wl}{4}=\frac{wl^2}{8} \quad \text{or} \quad W=wl/2$$

- For central load W, $U_1=\dfrac{W^2l^3}{96EI}$ (Refer Example 7.21)
- Figure 7.46 shows a simply supported beam of length l and carrying a uniformly distributed load w per unit length.

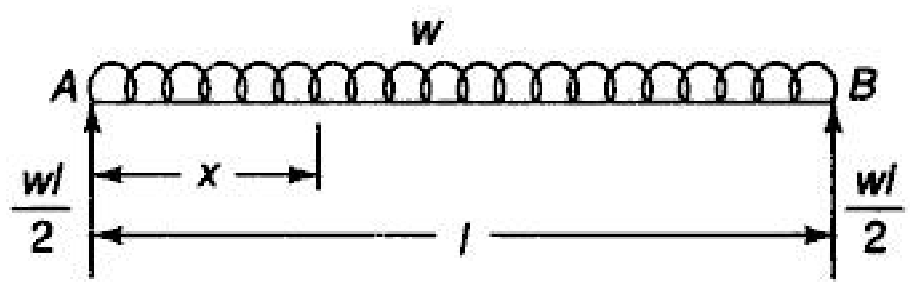

Fig. 7.46

$$M=\frac{wl}{2}.x-\frac{wx^2}{2}=\frac{wx}{2}(l-x)$$

$$U_2=\int_0^l\frac{M^2.dx}{2EI}=\int_0^l\frac{w^2}{4}x^2(l-x)^2\frac{dx}{2EI}=\frac{w^2}{8EI}\int_0^l(l^2x^2+x^4-2lx^3)dx$$

$$=\frac{w^2}{8EI}\left(\frac{l^2x^3}{3}+\frac{x^5}{5}-\frac{2lx^4}{4}\right)=\frac{w^2l^5}{240EI}$$

$$\therefore \quad \frac{U_1}{U_2}=\frac{W^2l^3}{96EI}\Big/\frac{w^2l^5}{240EI}=\frac{w^2.l^2l^3}{4\times 96EI}\Big/\frac{w^2l^5}{240EI}=\frac{5}{8}$$

Example 7.23 *Determine the maximum deflection of a simply supported beam of span l carrying a load of w per unit length using strain energy method.*

Solution Maximum deflection is at the midspan. Thus assume a concentrated load W at this point (Fig. 7.47).

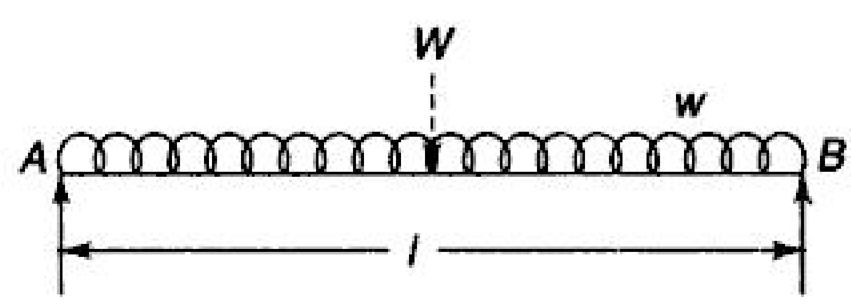

Fig. 7.47

$$R_a=R_b=\frac{W+wl}{2}$$

Bending moment at a distance x from

$$A=\frac{W+wl}{2}x-\frac{wx^2}{2}$$

$U=2\times$ strain energy in the half beam

$$=2\int_0^{l/2}\left(\frac{W+wl}{2}x-\frac{wx^2}{2}\right)^2\frac{dx}{2EI}=\frac{1}{2EI}\int_0^{l/2}\left(\frac{W+wl}{2}x-\frac{wx^2}{2}\right)^2dx$$

To find the deflection, differentiate the total strain energy with respect to W, i.e.

$$\delta=\frac{\partial U}{\partial W}=\frac{1}{EI}\int_0^{l/2}2\left(\frac{W+wl}{2}x-\frac{wx^2}{2}\right)\frac{x}{2}.dx$$

No.	*Simple (real) beam*	*Conjugate beam*
1.	Actual loading is the loading diagram.	Bending moment is the loading diagram.
2.	Bending moment diagram from loading diagram provides the bending moment at any section.	Bending moment diagram from loading diagram provides the deflection at any section.
3.	Shear force diagram provides the shear stress at a section.	Shear force diagram provides the slope at a section.
4.	Shear force and bending moment at the fixed end of a cantilever exist.	S.F. and B.M. at fixed end will provide some values of slope and deflection which are not feasible. Thus a fixed end is transformed into a free end to obtain S.F. as well as B.M. as zero.
5.	Shear force and bending moment at the free end of a cantilever are zero.	S.F. and B.M. at the free end will provide zero slope and deflection which are not feasible. Thus a free end is transformed into a fixed end.
6.	Bending moment at the supports of simply supported beam is zero.	Deflection at the supports is zero. So, end conditions remain same.
7.	Shear force moment at the supports of simply supported beam exists.	Slope at the supports exists. So, end conditions remain same.
8.	An intermediate support has same slope on both sides. Also, an intermediate support has no deflection.	To have same shear force (slope) on both sides and zero bending moment (deflection), it is transformed into an intermediate hinge.
9.	A hinge support has same shear force and zero bending moment.	It is transformed into an intermediate support.

Example 7.28 *Find expressions for the central deflection and the slope at the ends of a beam simply supported at the ends by conjugate beam method.*

Solution Figure 7.48*a* shows the actual loading diagram. As maximum bending moment at the centre is *Wl*/4, in the *M/EI* diagram the same is shown as *Wl*/4*EI* in Fig. 7.48*b*.

Now, in the conjugate beam method, this diagram is to be considered as loading diagram and a new bending moment

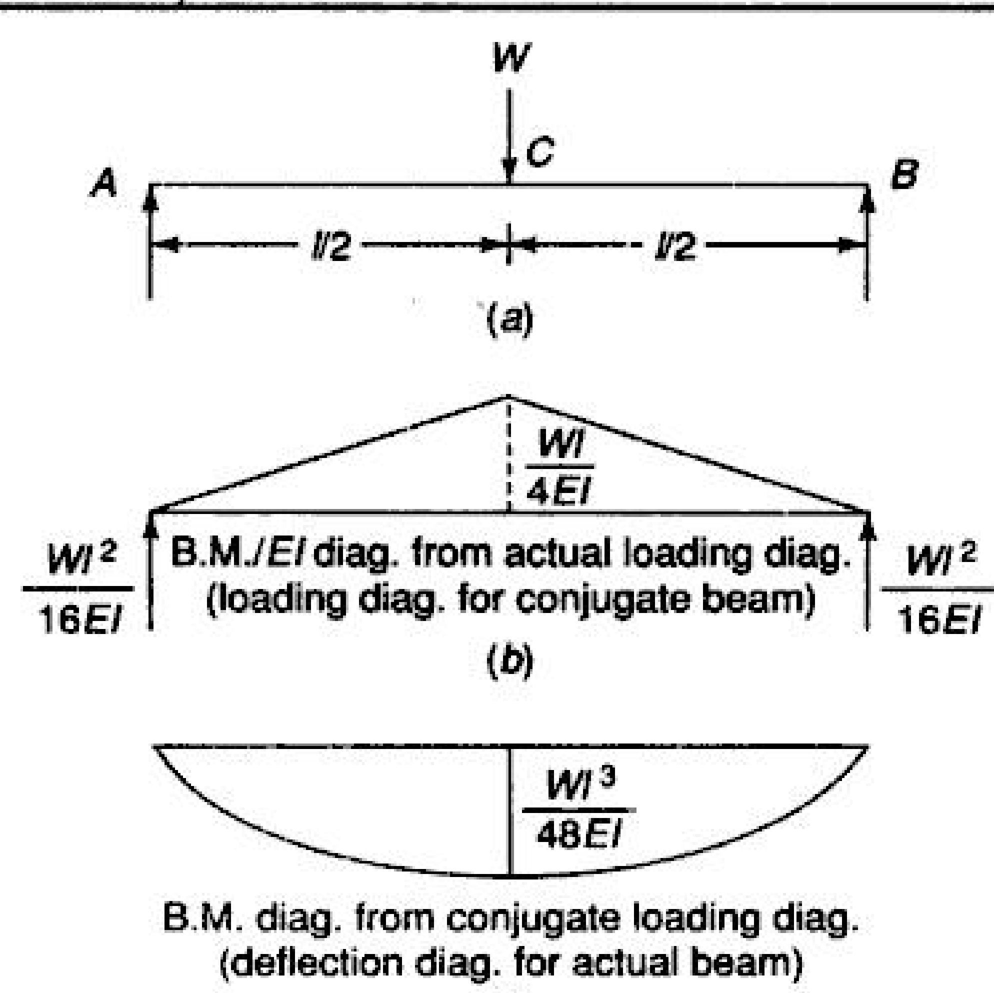

Fig. 7.48

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Slope at the free end = shearing force at B

for conjugate beam = $\frac{1}{3.}l.\frac{wl^2}{2}.\frac{1}{EI}=\frac{Wl^3}{6EI}$

Deflection at the free end = bending moment

at B for conjugate beam = $\frac{wl^3}{6EI}\frac{3l}{4}=\frac{wl^4}{8EI}$

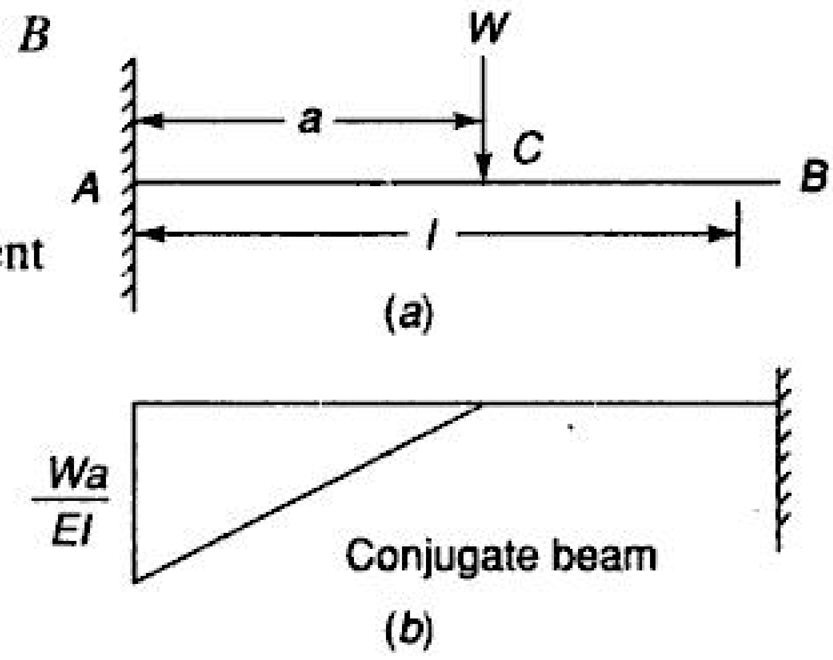

Fig. 7.52

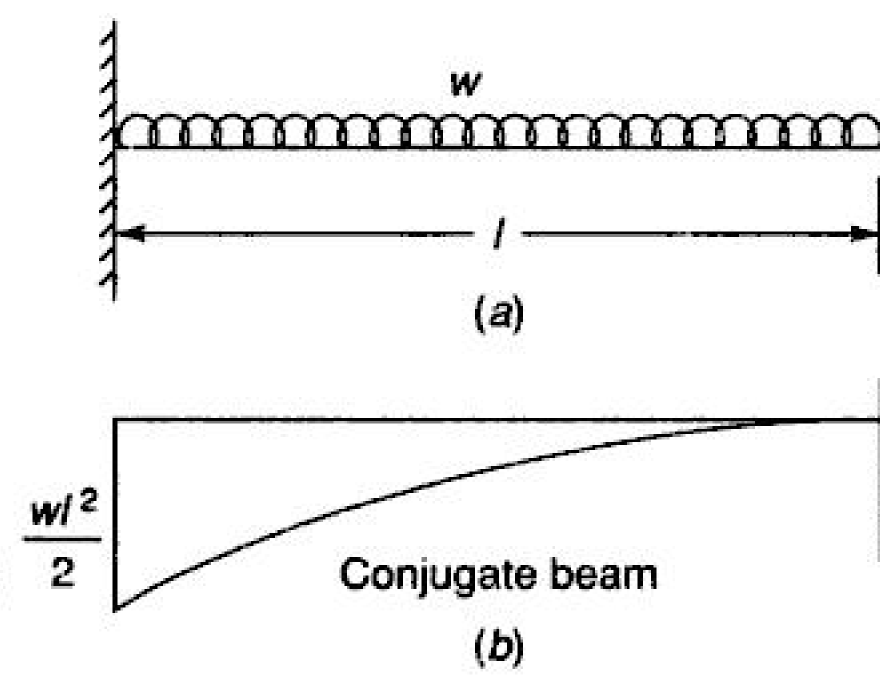

Fig. 7.53

Example 7.34 *A beam is loaded as shown in Fig. 7.54a. Determine the slopes at the load and the support points and the deflections at the load points using conjugate beam method.*

Solution To find reaction at the supports, take moments about B,

$$R_a.l = 2W.\frac{l}{2} - W.\frac{l}{4} \quad \text{or} \quad R_a = \frac{3W}{4}$$

and $R_b = 3W - \frac{3W}{4} = \frac{9W}{4}$

Bending moment at $C = \frac{3W}{4}.\frac{l}{2} = \frac{3Wl}{8}$;

Bending moment at $B = -W.\frac{l}{4} = \frac{Wl}{4}$

Point of contraflexture in CB: Let this point be at a distance x from C,

$$\frac{3W}{4}\left(\frac{l}{2}+x\right) - 2Wx = 0 \quad \text{or} \quad \frac{3l}{8}+\frac{3x}{4}-2x=0 \quad \text{or} \quad x=\frac{3l}{10}$$

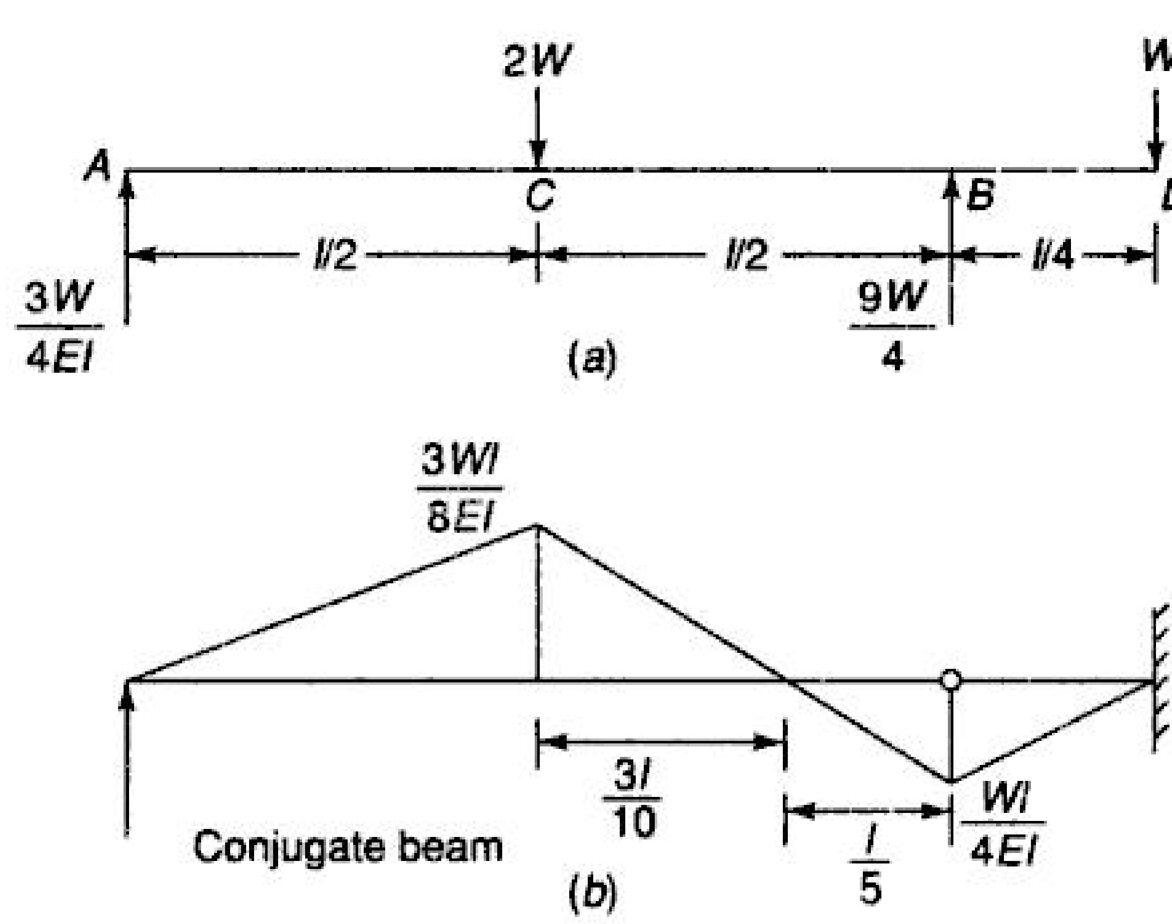

Fig. 7.54

moment of this beam gives the deflection curve of the actual loading and the new shear force diagram provides the slope.

- Maxwell's theorem of reciprocal displacements states that the deflection of any point *P* resulting from application of a load at any other point *Q* is the same as the deflection of *Q* resulting from the application of the same load at *P*.
- Betti's theorem of reciprocal deflections states that in an elastic system, the external work done by a force acting at *P* during the deflections caused by another force at *Q* is equal to the external work done by the force at *Q* during the deflections caused by the force at *P*.

Review Questions

1. Establish the governing differential equation of beams. What are its limitations?
2. What is Macaulay's method of beam deflection analysis? What are its advantages over the direct integration method?
3. State and prove the moment–area theorem.
4. State and develop the analogies between the real beam and the conjugate beam.
5. State and prove Castigliano's first theorem.
6. Deduce the expressions for deflection by the energy method.
7. State and prove Maxwell's reciprocal deflection theorem.
8. What is Betti's theorem of reciprocal deflection?
9. The rate of loading on a simply supported beam of length l is $p \sin \pi x/l$ where x is the distance from one end. Show that the reactions at the supports are pl/π and the maximum bending moment is pl^2/π^2.

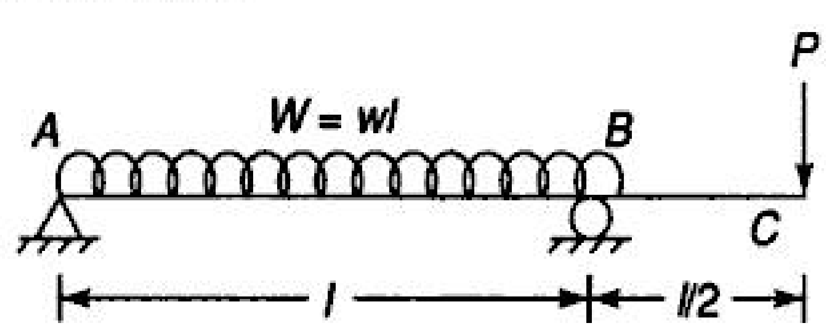

Fig. 7.56

10. A simply supported beam with an overhang is loaded as shown in Fig. 7.52. Find the ratio of W/P to make the deflection at the free end equal to zero. (6)
11. Determine the maximum deflection of a simply supported beam of 5-m length and carrying a uniformly distributed load from zero at the ends to 8 kN/m at the centre. $EI = 2$ MN.m^2. (20.8 mm)
12. A simply supported horizontal beam carries a load which varies from 20 kN at one end to 50 kN at the other. Determine the central deflection if the span is 10 m and the width is 420 mm. The bending stress is limited to 84 MPa. $E = 210$ GPa. (25 mm)
13. A simply supported beam has a span of 15 m and carries two point loads of 4 kN and 9 kN at 6 m and 10 m respectively from one end. Find the deflection under each load and the maximum deflection. $E = 200$ GPa and $I = 400 \times 10^6$ mm^4. (9.39 mm, 8.99 mm; 10.48 mm at 7.79 m)
14. A beam *AB* of 6-m span is simply supported at the ends. It carries a concentrated load of 6 kN at a distance of 6 m from the left-hand end support and a uniformly

Similarly, if the deflection of one end relative to other is zero, the moments of areas of the bending moment diagram about an end are zero, i.e.

Moment of area of free moment diagram = Moment of area of fixed moment diagram

or $$A_1\bar{x}_1 = A_2\bar{x}_2 \tag{8.2}$$

The area to be considered may be broken into parts to obtain convenient rectangles, triangles and parabolas.

Total reactions at the ends,

$$R_a = R_1 + R = R_1 + \frac{M_a - M_b}{l}$$

and $$R_b = R_2 - R = R_2 - \frac{M_a - M_b}{l}$$

Example 8.1 *Determine the maximum bending moment and the deflection of a beam of length l and flexural rigidity EI. The beam is fixed horizontally at both ends and carries a concentrated load W at the midspan.*

Solution The beam is shown in Fig. 8.2*a*.

Due to symmetry, fixing moment $M_a = M_b = M$ (say)

The free moment diagram is a triangle with maximum ordinate *Wl*/4 as shown in Fig. 8.2*b*.

- As the slope at *A* is equal to slope at *B* = 0, net area of the moment diagram must be zero, i.e.

$$\frac{1}{2}.\frac{Wl}{4}.l = Ml \quad \text{or} \quad M = \frac{Wl}{8}$$

The combined bending moment diagram is shown in Fig. 8.2*c*. The maximum bending moment is *Wl*/8 hogging (at the ends) and sagging (*wl*/4 – *wl*/8, at the centre).

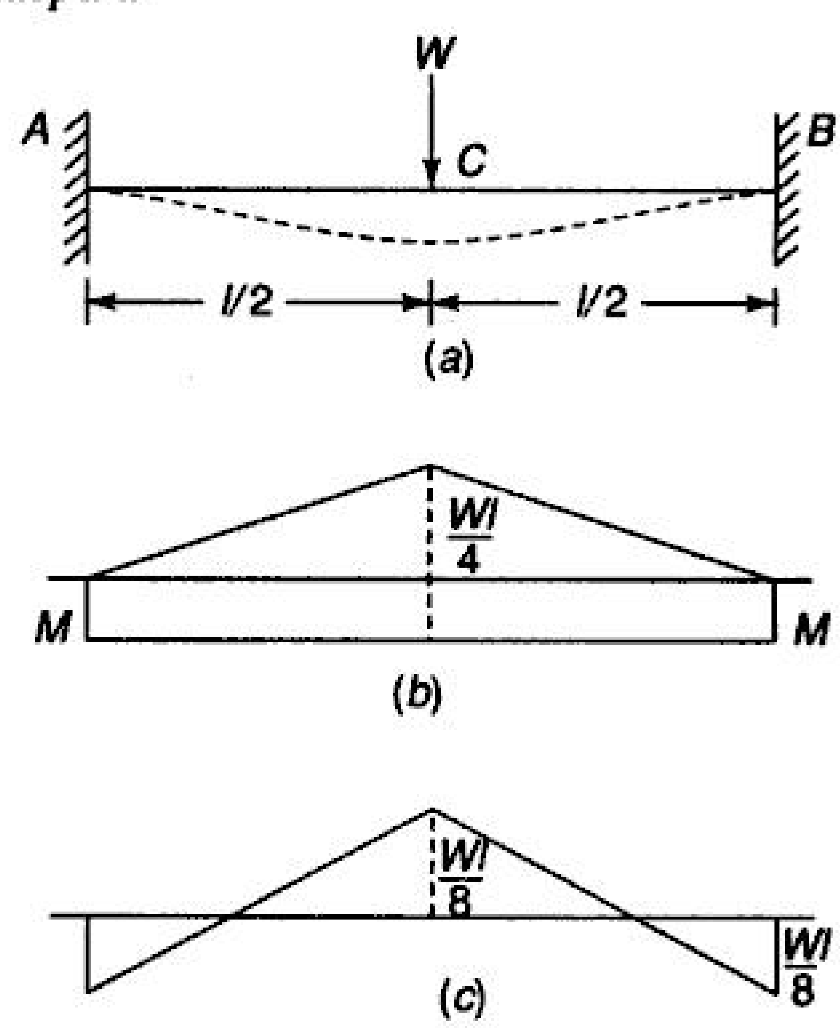

Fig. 8.2

- To find the deflection of *C* relative to *A*, Take moments of the areas of the bending moment diagram between *A* and *C* about *A*,

$$y = \frac{1}{EI}\left[\left(\frac{1}{2}.\frac{Wl}{8}.\frac{l}{4}\right)\left(\frac{2}{3}.\frac{l}{2}\right) - \left(\frac{Wl}{8}.\frac{l}{2}\right).\frac{l}{4}\right] = \frac{Wl^3}{192EI} \text{ (From Fig. 8.2}b\text{)}$$

Example 8.2 *Determine the maximum bending moment and the deflection of a beam of length l and flexural rigidity EI. The beam is fixed horizontally at both ends and carries a uniformly distributed load w over the whole span.*

Solution The beam is shown in Fig. 8.3*a*.

Due to symmetry, fixing moment $M_a = M_b = M$ (say)

Thus $\frac{M_a + M_b}{2} l = \frac{wl^3}{24}$ or $M_a + M_b = \frac{wl^2}{12}$ (i)

- $A_1\bar{x}_i = A_2\bar{x}_2$

$$A_1\bar{x}_1 = \int_0^{l/2} M_x.x + \left(\frac{1}{2}.\frac{wl^2}{16}.\frac{l}{2}\right).\left(\frac{l}{2}+\frac{1}{3}.\frac{l}{2}\right) = \int_0^{l/2}\left(\frac{3wl}{8}x^2 - \frac{wx^3}{2}\right)dx + \frac{wl^4}{96}$$

$$\left(\frac{3wl}{8}.\frac{x^3}{3} - \frac{wx^4}{8}\right)_0^{l/2} + \frac{wl^4}{96} = \frac{7wl^4}{384}$$

Thus $\frac{1}{2}.M_a.l.\frac{l}{3} + \frac{1}{2}.M_b.l.\frac{2l}{3} = \frac{7wl^4}{384}$

or $(M_a + 2M_b)\frac{l^2}{6} = \frac{7wl^4}{384}$ or $M_a + 2M_b = \frac{7wl^2}{64}$ (ii)

Subtract (i) from (ii),

$$M_b = \frac{7wl^2}{64} - \frac{wl^2}{12} = \frac{5wl^2}{192} \text{ and } M_a = \frac{wl^2}{12} - \frac{5wl^2}{192} = \frac{11wl^2}{192}$$

Bending moment diagram has been shown in Fig. 8.7*c*.

Reactions at the ends,

$$R_a = R_1 + \frac{M_a - M_b}{l} = \frac{3wl}{8} + \frac{6wl}{192} = \frac{13wl}{32}$$

and $$R_b = R_2 - \frac{M_a - M_b}{l} = \frac{wl}{8} - \frac{6wl}{192} = \frac{3wl}{32}$$

Shear force diagram has been shown in Fig. 8.7*d*.

Example 8.6 *A beam of 18-m span is fixed horizontally at the ends. A downward point load of 18 kN acts on the beam at a distance of 6 m and an upward force of 12 kN at a distance of 9 m from the left-hand end. Find the end reactions and the fixing moments. Also draw the bending moment and shear force diagrams.*

Solution The beam is shown in Fig. 8.8*a*. Let the fixing moments be M_a and M_b at the two ends.

- $A_1 = A_2$

  Take moments about end *B*,

  $R_1 \times 18 - 18 \times 12 + 12 \times 9 = 0$ or $R_1 = 6$ kN and $R_2 = 18 - 12 - 6 = 0$

  Bending moment diagrams for the free and fixing moments are shown in Fig. 8.2*b*.

  Thus $\frac{M_a + M_b}{2} \times 18 = \frac{36 \times 9}{2}$ or $M_a + M_b = 18$ (i)

- $A_1\bar{x}_1 = A_2\bar{x}_2$

  $(M_a + 2M_b)\frac{18^2}{6} = \frac{6 \times 36}{2} \times 4 + \frac{3 \times 36}{2} \times 7$ or $(M_a + 2M_b) = 15$ (ii)

- Integrating again, $EIy = -M_a.\frac{x^2}{2} + \frac{wlx^3}{12} - \frac{wx^4}{24} + 0$ ...(Constant B will be zero)

  Also, At $x = l$, $y' = 0$ and $y = 0$

$$\therefore \quad 0 = -M_a.l + \frac{wl^3}{4} - \frac{wl^3}{6} \quad \text{or} \quad M_a = \frac{wl^2}{12} \quad \text{and} \quad M_b = M_a = \frac{wl^2}{12}$$

Maximum deflection at midspan, $EIy = -\frac{wl^2}{12}.\frac{l^2}{8} + \frac{wl^4}{96} - \frac{wl^4}{384}$ or $y = -\frac{Wl^4}{384EI}$

Example 8.10 *A beam has its ends fixed horizontally at the same level. The beam is of length l and carries a load W at a distance a from one end and b from the other end. Determine the fixing moments at the ends.*

Solution Let the load be at a distance a from the left-hand fixed end (Fig. 8.12)

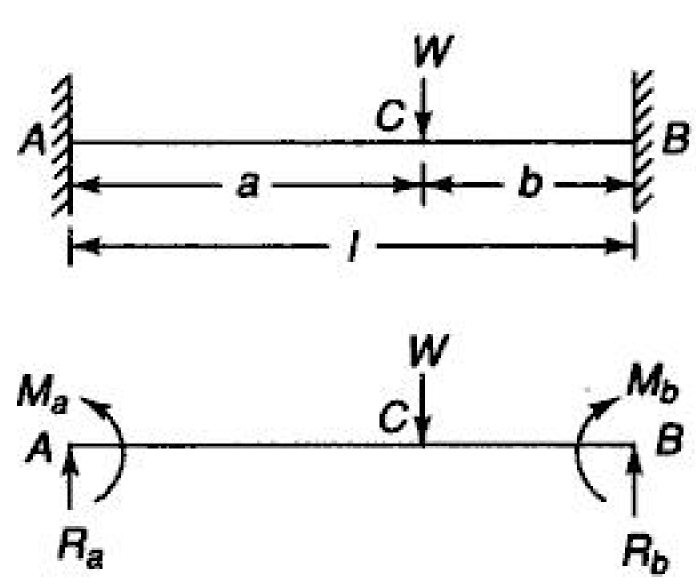

Fig. 8.12

- $EIy'' = -M_a + R_a x \big| - W(x-a)$
- Integrating, $EIy' = -M_a.x + R_a\frac{x^2}{2} + 0 \Big| - \frac{W}{2}(x-a)^2$

  ...(At $x = 0$, $y' = 0$, $\therefore A = 0$)
- Integrating again,

$$EIy = -M_a.\frac{x^2}{2} + R_a\frac{x^3}{6} + 0 \Big| - \frac{W}{6}(x-a)^3$$

...($B = 0$)

Also, At $x = l$, $y' = 0$ and $y = 0$

$$\therefore \quad 0 = -M_a.l + R_a\frac{l^2}{2} - \frac{W}{2}(l-a)^2 \quad \text{or} \quad 2M_a.l = R_a l^2 - Wb^2 \qquad \text{(i)}$$

$$\text{and } 0 = -M_a.\frac{l^2}{2} + R_a\frac{l^3}{6} - \frac{W}{6}(l-a)^3 \quad \text{or} \quad 3M_a.l^2 = R_a l^3 - Wb^3 \qquad \text{(ii)}$$

Multiplying (i) by l and subtracting from (ii),

$$M_a l^2 = Wb^2(l-b) = Wab^2 \quad \text{or} \quad M_a = \frac{Wab^2}{l^2}$$

From (i), $2\frac{Wab^2}{l^2}l = R_a l^2 - Wb^2$

$$\text{or} \quad R_a = \frac{Wb^2}{l^3}(2a + l) = \frac{Wb^2}{l^3}(2a + a + b) = \frac{Wb^2}{l^3}(3a + b)$$

- Bending moment at right fixed end,

$$-M_b = -\frac{Wab^2}{l^2} + \frac{Wb^2}{l^3}(3a+b)l - W(l-a)$$

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Solution The load diagram is shown in Fig. 8.16*a*.

For *AB*, $M_{max} = \frac{3\times5^2}{8} = 9.325$ kN.m; For *BC*, $M_{max} = \frac{2\times10^2}{8} = 25$ kN.m;

For *CD*, $M_{max} = \frac{4\times3^2}{8} = 4.5$ kN.m

Also, $M_a = M_d = 0$

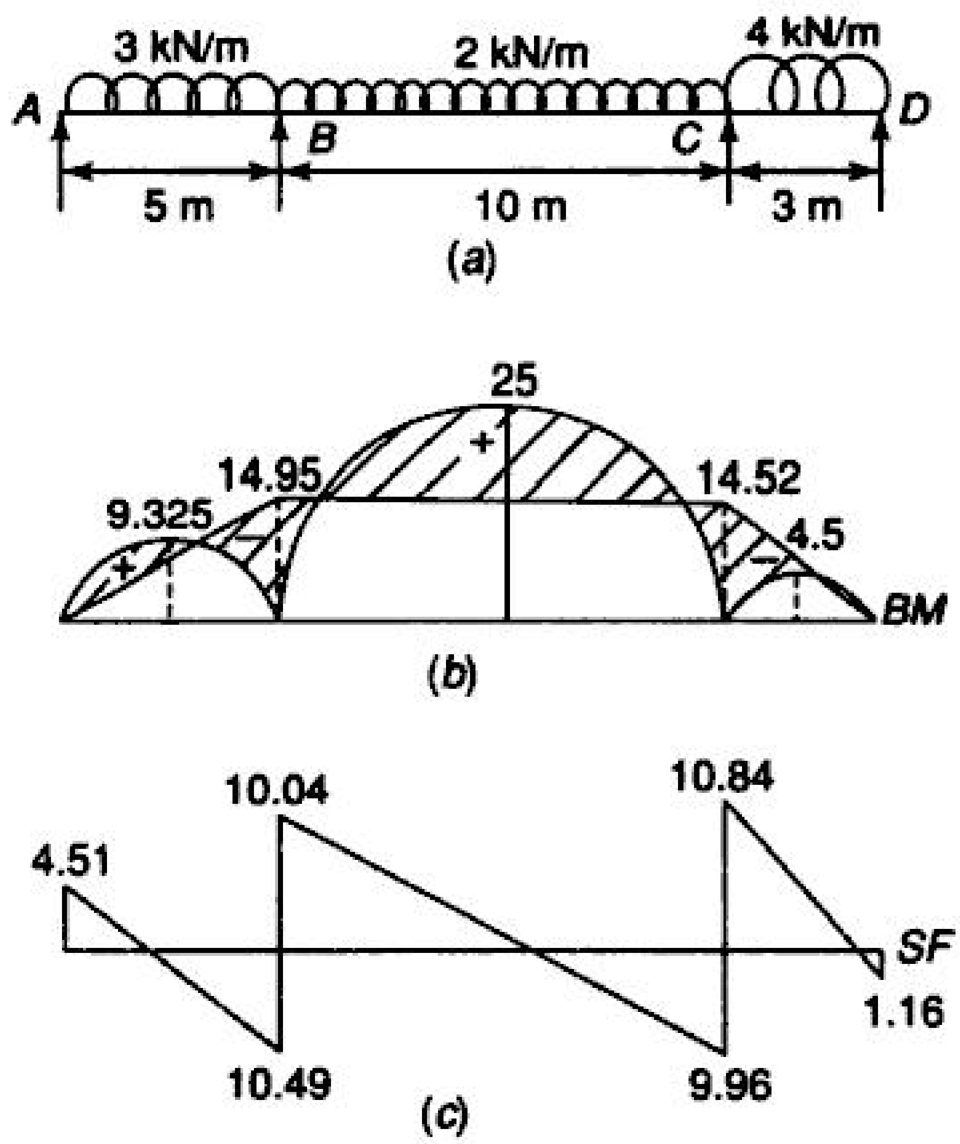

Fig. 8.16

Apply the three-moment equation for uniformly distributed loads to the spans *AB* and *BC*,

$$M_a \times 5 + 2M_b(5+10) + M_c \times 10 = -\frac{3\times5^3}{4} - \frac{2\times10^3}{4}$$

or $30M_b + 10M_c = -593.75$

or $M_b + 0.333M_c = -19.79$ (i)

Apply the three-moment equation for uniformly distributed loads to the spans *BC* and *CD*,

$$M_b \times 10 + 2M_c(10+3) + M_d \times 3 = -\frac{2\times10^3}{4} - \frac{4\times3^3}{4}$$

or $10M_b + 26M_c = -527$

or $M_b + 2.6M_c = -52.7$ (ii)

Subtracting (i) from (ii),

$2.267M_c = -32.91$ or $M_c = -14.52$ kN.m and $M_b = -14.95$ kN.m

To find R_a, take moments about *B*,

$R_a \times 5 - 3\times5\times2.5 = -14.95$ or $R_a = 4.51$ kN

To find R_b, consider bending moment at *C*,

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- *At D*: Stiffness factor for *CD*, $s_{dc} = \dfrac{4EI}{6} = \dfrac{2EI}{3}$ (as the beam is fixed at *A*)

  and for *DE*, $s_{de} = \dfrac{3EI}{6} = \dfrac{EI}{2}$ (as the beam is continuous at *C*)

  Distribution factor for *CD*, $k_{dc} = \dfrac{2EI/3}{2EI/3 + EI/2} = \dfrac{4}{7}$

  Distribution factor for *DE*, $k_{de} = \dfrac{EI/2}{2EI/3 + EI/2} = \dfrac{3}{7}$

	A	B	B	C	C	D	D	E
Distribution factors		1/2	1/2	1/2	1/2	4/7	3/7	
Fixed end moments	–3	3	–6	6	–6	6	–3	3
Release *E*								–3
Carry over							–1.5	
Net moments	–3	3	–6	6	–6	6	–4.5	0
Distribute		1.5	1.5	0	0	–0.86	–0.64	
Carry over	0.75			0.75	–0.43			
Distribute				–0.16	–0.16			
Carry over			–0.08			–0.08		
Distribute		0.04	0.04			0.046	0.034	
Final moments	–2.25	4.54	4.54	6.59	–6.59	5.106	–5.106	0

Bending moment diagram:

For span *AB*, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{1 \times 6^2}{8} = 4.5$

For span *BC*, $M_{\max} = \dfrac{wl^2}{8} + \dfrac{Wl}{4} = \dfrac{1 \times 6^2}{8} + \dfrac{4 \times 6}{4} = 10.5$ kN.m

For span *CD*, $M_{\max} = \dfrac{Wl}{4} = \dfrac{8 \times 6}{4} = 12$ kN.m

For span *DE*, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{1 \times 6^2}{8} = 4.5$ kN.m

Now, the bending moment diagram can be completed as shown in Fig. 8.29*b*.

- *Shear Force diagram*

  To find reaction R_e, taking moments about *D*,

  $R_e \times 6 - 6 \times 3 = -5.108$ or $R_e = 2.15$ kN

  Taking moments about *C*,

  $2.15 \times 12 - 6 \times 9 + R_d \times 6 - 8 \times 3 = -6.588$ or $R_d = 7.6$ kN

  Taking moments about *B*,

  $2.15 \times 18 - 6 \times 15 + 7.6 \times 12 - 8 \times 9 + R_c \times 6 - 4 \times 3 - 6 \times 3 = -4.541$

  or $R_c = 9.59$ kN

  Taking moments about *B*,

  $R_a \times 6 - 2.229 - 6 \times 3 = -4.541$ or $R_a = 2.61$ kN

  $R_b = 12 + 4 + 8 + 6 - 2.15 - 7.6 - 9.59 - 2.61 = 8.05$ kN

Consider an element at a distance y from the neutral axis.
Strain in the element

$$= \frac{AC - AB}{AB} = \frac{(R' + y)(\theta + \delta\theta) - (R + y)\theta}{(R + y)\theta}$$

$$= \frac{R'(\theta + \delta\theta) + y\theta + y\delta\theta - R\theta - y\theta}{(R + y)\theta}$$

$$= \frac{R'(\theta + \delta\theta) - R\theta + y\delta\theta}{(R + y)\theta}$$

As length of the neutral axis remains constant,

$$R'(\theta + \delta\theta) = R\theta$$

and

$$\delta\theta = \frac{R - R'}{R'}\theta$$

Strain

$$= \frac{y\delta\theta}{(R + y)\theta} = \frac{y(R - R')\theta}{(R + y)\theta . R'}$$

If y is neglected compared to R, Strain = $y\left(\frac{1}{R'} - \frac{1}{R}\right)$

Normal stress (neglecting the lateral stress),

$$\sigma = Ey\left(\frac{1}{R'} - \frac{1}{R}\right) \quad \text{(i)}$$

As total normal stress is to be zero,

$$\int \sigma \cdot dA = E\left(\frac{1}{R'} - \frac{1}{R}\right)\int y \cdot dA = 0$$

or

$$\int y \cdot dA = 0$$

which indicates that the neutral axis passes through the centroid of the section.
Moment of resistance,

$$M = \int \sigma y dA = E\left(\frac{1}{R'} - \frac{1}{R}\right)\int y^2 \cdot dA = EI\left(\frac{1}{R'} - \frac{1}{R}\right) \quad \text{(ii)}$$

From (i) and (ii),

$$\frac{M}{I} = \frac{\sigma}{y} = E\left(\frac{1}{R'} - \frac{1}{R}\right) \quad (9.1)$$

Strain energy of a small length δs along the neutral axis under the action of bending moment M,

$$\delta U = \frac{1}{2}M.\delta\theta = \frac{1}{2}M \cdot \frac{R - R'}{R'}\theta$$

$$= \frac{1}{2}(M\theta)R\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{1}{2}M\delta s\frac{M}{EI} = \frac{M^2}{2EI}\delta s \quad (9.2)$$

Example 9.1 *A piston ring is required to keep its outer surface circular in the stressed as well as in the unstressed condition when a uniform radial pressure exerted is uniform. Express the variation of thickness of its surface along the angular direction*

Then $\int \frac{y}{1 + y/R}\, dA = -\frac{1}{R}\int \frac{y}{1 + y/R}\, dA = -\frac{1}{R} Ap^2$ (iv)

Eq. 9.5 becomes

$$F = EA\left[\varepsilon' - (1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{p^2}{R}\right] \tag{9.7}$$

As transverse plane sections before bending remain plane after bending, $F = 0$

$$\varepsilon' = (1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{p^2}{R}$$

Inserting this value of ε' in Eq. 9.4,

Now,
$$\sigma = E\left[(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{p^2}{R} + (1 + \varepsilon')\frac{y}{1 + y/R}\left(\frac{1}{R'} - \frac{1}{R}\right)\right]$$

$$= E(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{1}{R}\left[p^2 + \frac{yR}{1 + y/R}\right]$$

But from Eq. 9.6, $M = E(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2$ (using iii) (9.8)

or $E(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{M}{Ap^2}$

$$\sigma = \frac{M}{Ap^2}\cdot\frac{1}{R}\left[p^2 + \frac{yR^2}{R + y}\right] = \frac{M}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R + y}\right] \tag{9.9}$$

At inside of the centroidal axis, y is negative and thus,

$$\sigma = \frac{M}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R - y}\right]$$

The position of the neutral axis can be found from the fact that at the neutral axis, $\sigma = 0$,

$$\sigma = \frac{M}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R + y}\right] = 0$$

or $yR^2 = -Rp^2 - yp^2$ or $y = \frac{-Rp^2}{R^2 + p^2}$ (9.10)

p^2 for different sections can be evaluated easily by the following simplified expression:

$$p^2 = \frac{1}{A}\int \frac{Ry^2}{R + y}\, dA = \frac{1}{A}\int\left(Ry - \frac{R^2 y}{R + y}\right)dA$$

$$= \frac{1}{A}\int\left[Ry - R^2\left(1 - \frac{R}{R + y}\right)\right]dA$$

$$= \frac{R}{A}\left[\int y\,dA - \int R\,dA + \int \frac{R^2}{R + y}\,dA\right]$$

$$= -\frac{W}{A}\cdot\frac{R^2}{p^2}\,\frac{y}{R+y} = -\frac{80\,000}{4000}\cdot\frac{202^2}{1839.5}\cdot\frac{88}{202+88}$$

$= -443.64 \times 0.3034 = -134.6$ MPa (compressive)

Stress at inside face (Q); $y = d = 52$ mm

$$\sigma_i = \frac{W}{A} - \frac{WR}{AR}\left[1 - \frac{R^2}{p^2}\,\frac{y}{R-y}\right] = \frac{W}{A}\cdot\frac{R^2}{p^2}\cdot\frac{y}{R-y}$$

$$= 443.64 \times \frac{52}{202-52} = 153.8 \text{ MPa}$$ (tensile)

Example 9.5 *An open ring has an I-section as shown in Fig. 9.13. It is subjected to a load of 50 kN. Find the stresses at points P and Q.*

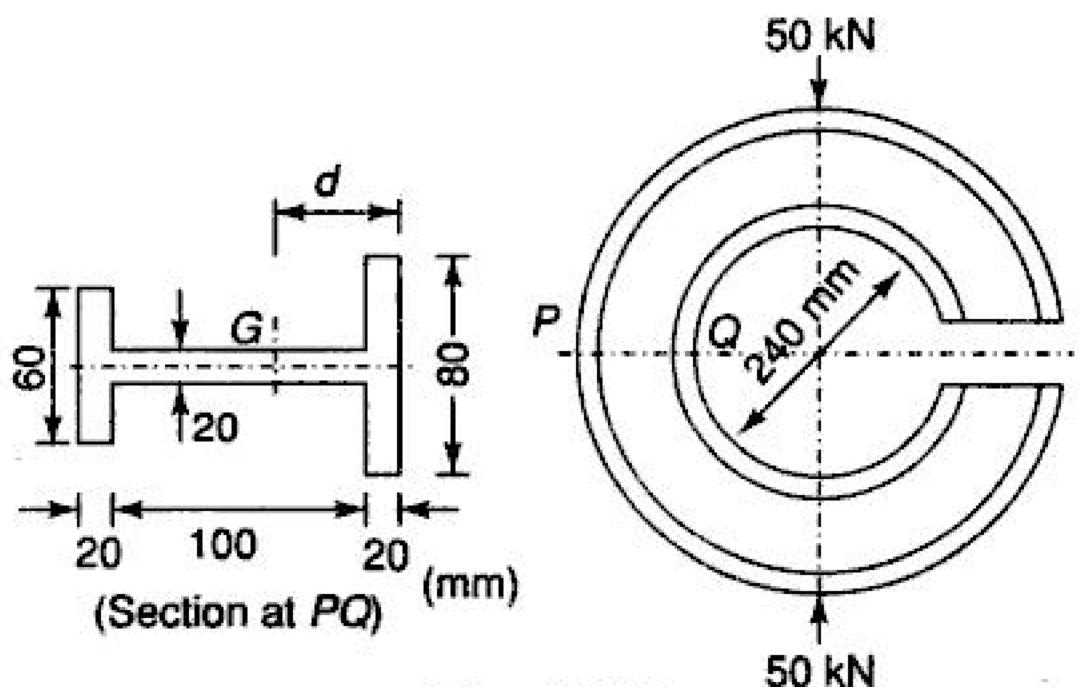

Fig. 9.13

Solution

$$A = 80 \times 20 + 100 \times 20 + 60 \times 20 = 4800 \text{ mm}^2.$$

$$d = \frac{80 \times 20 \times 10 + 100 \times 20 \times 70 + 60 \times 20 \times 130}{4800} = 65 \text{ mm}$$

$R = 120 + 65 = 185$ mm

$R_1 = 120$ mm; $R_2 = 140$ mm, $R_3 = 120 + 20 + 100 = 240$ mm

and $R_4 = 240 + 20 = 260$ mm

$$p^2 = \frac{R^3}{A}\left[b_1 \ln\frac{R_2}{R_1} + t \ln\frac{R_3}{R_2} + b_2 \ln\frac{R_4}{R_3}\right] - R^2$$

$$= \frac{185^3}{4800}\left[80 \ln\frac{140}{120} + 20 \ln\frac{240}{140} + 60 \ln\frac{260}{240}\right] - 185^2 = 2597 \text{ mm}^2$$

Stress at outside face (P)

$$\sigma_o = -\frac{W}{A} + \frac{WR}{AR}\left[1 + \frac{R^2}{p^2}\,\frac{y}{R+y}\right]$$

$$= \frac{W}{A}\cdot\frac{R^2}{p^2}\,\frac{y}{R+y} = \frac{50\,000}{4800}\cdot\frac{185^2}{2597}\,\frac{75}{185+75}$$

$= 137.28 \times 0.2885 = 39.6$ MPa (tensile)

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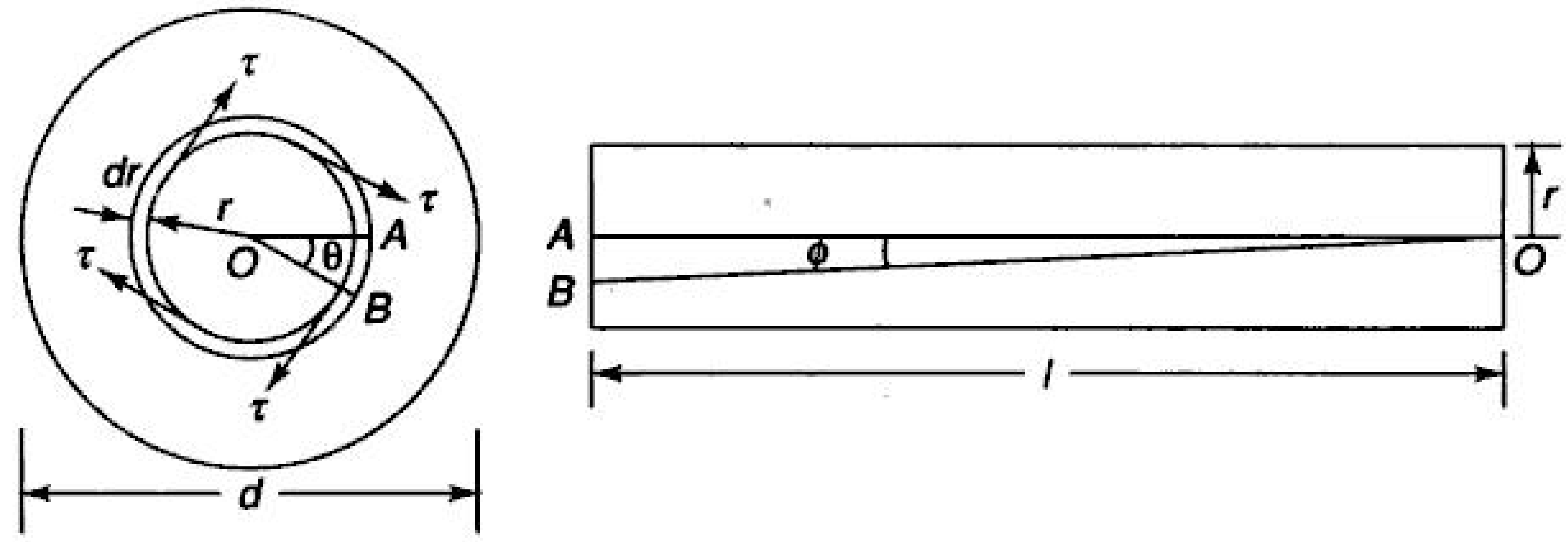

Fig. 10.1

Let θ be the angle of twist of a cross-section relative to another cross-section at distance l apart. Then

$$\text{Arc } AB = r\theta = l\phi = l.\frac{\tau}{G} \qquad \text{........}(G = \tau/\varphi)$$

or
$$\frac{\tau}{r} = \frac{G\theta}{l} \qquad \text{(i)}$$

Now, tangential force on the element = $\tau(2\pi r.\ dr)$
Moment of the tangential force on the element = $\tau(2\pi r.\ dr)r$
Sum of moments for the whole shaft,

$$T = \int \tau(2\pi r.dr)r$$

$$= \int \frac{G\theta}{l} r.(2\pi r.dr)r = \frac{G\theta}{l}\int (2\pi r.dr)r^2$$

But $\int (2\pi r.dr)r^2$ is the polar moment of inertia of the shaft,

Thus
$$T = \frac{G\theta}{l}.J \qquad \text{(ii)}$$

From (i) and (ii),
$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l} \qquad (10.1)$$

This shows that for a given shaft, shear stress is proportional to radius.

- ***Solid shaft*** For a solid shaft, $J = \dfrac{\pi d^4}{32}$

  $\therefore$ Maximum shear stress, $\tau = \dfrac{T.d/2}{\pi d^4/32} = \dfrac{16T}{\pi d^3}$ at the outer surface (10.2)

- ***Hollow shaft*** If D and d are the outer and inner diameters respectively of the hollow shaft, then

$$J = \frac{\pi(D^4 - d^4)}{32}$$

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has a ratio of 0.9 from the inner to outer diameter. Also determine the error in the angle of twist.

Solution

- By Bredt–Batho theory,

$$T = 2\tau.t.A \quad \text{or} \quad T = 2\tau\left(\frac{D-d}{2}\right).\frac{\pi}{4}\left(\frac{D+d}{2}\right)^2$$

$$= 2\tau.\frac{\pi}{16}.D\left(1-\frac{d}{D}\right).D^2\left(1+\frac{d}{D}\right)^2$$

$$= 2\tau.\frac{\pi}{16}.D(1-0.9).D^2(1+0.9)^2 \quad \text{or} \quad \tau = \frac{16T}{1.134D^3}$$

By normal theory, $\tau' = \dfrac{16TD}{\pi(D^4-d^4)} = \dfrac{16TD}{\pi D^4(1-0.9^4)} = \dfrac{16T}{1.08D^3}$

Thus, ratio $\dfrac{\tau}{\tau'} = \dfrac{1.08}{1.134} = 0.952$

- By Bredt–Batho theory,

$$\theta = \frac{lTz}{4GA^2t} = \frac{lT}{4G}.\pi\left(\frac{D+d}{2}\right).\frac{1}{[(\pi/4)\{(D+d)/2\}^2]^2}.\frac{1}{(D-d)/2}$$

$$= \frac{lT}{4G}.\frac{\pi D}{2}(1+0.9).\frac{16}{\pi^2}\frac{16}{D^4(1+0.9)^4}.\frac{1}{D}\frac{2}{(1-0.9)}$$

$$= 29.71\times\frac{Tl}{GD^4}$$

By shaft theory, $\theta' = \dfrac{l}{G}.\dfrac{32T}{\pi(D^4-d^4)} = \dfrac{l}{G}.\dfrac{32T}{\pi D^4(1-0.9^4)} = 29.62\times\dfrac{Tl}{GD^4}$

Thus, ratio $\dfrac{\theta'}{\theta} = \dfrac{29.62}{29.71} = 0.997$

10.9 THIN-WALLED SECTIONS

Consider a thin-walled twin-celled section shown in Fig. 10.14. Let ADC be of uniform thickness t_1 and stress τ_1, CBA of uniform thickness t_2 and stress τ_2, and AC of uniform thickness t_3 and stress τ_3. Also let A_1 and A_2 be the mean areas of the two cells. It may be assumed that the direction of shear flow in AC is downwards.

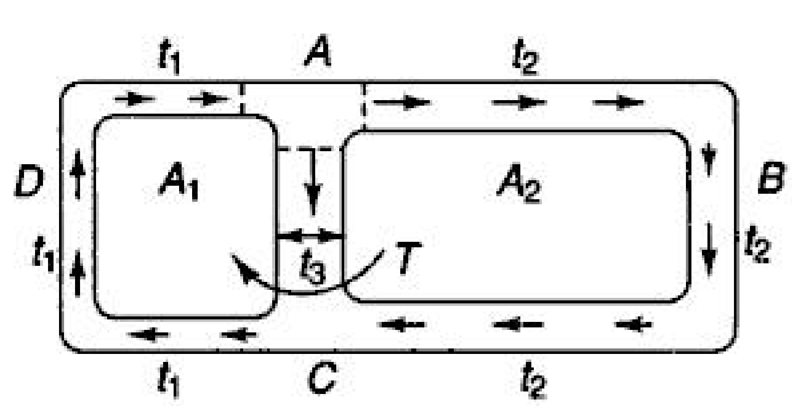
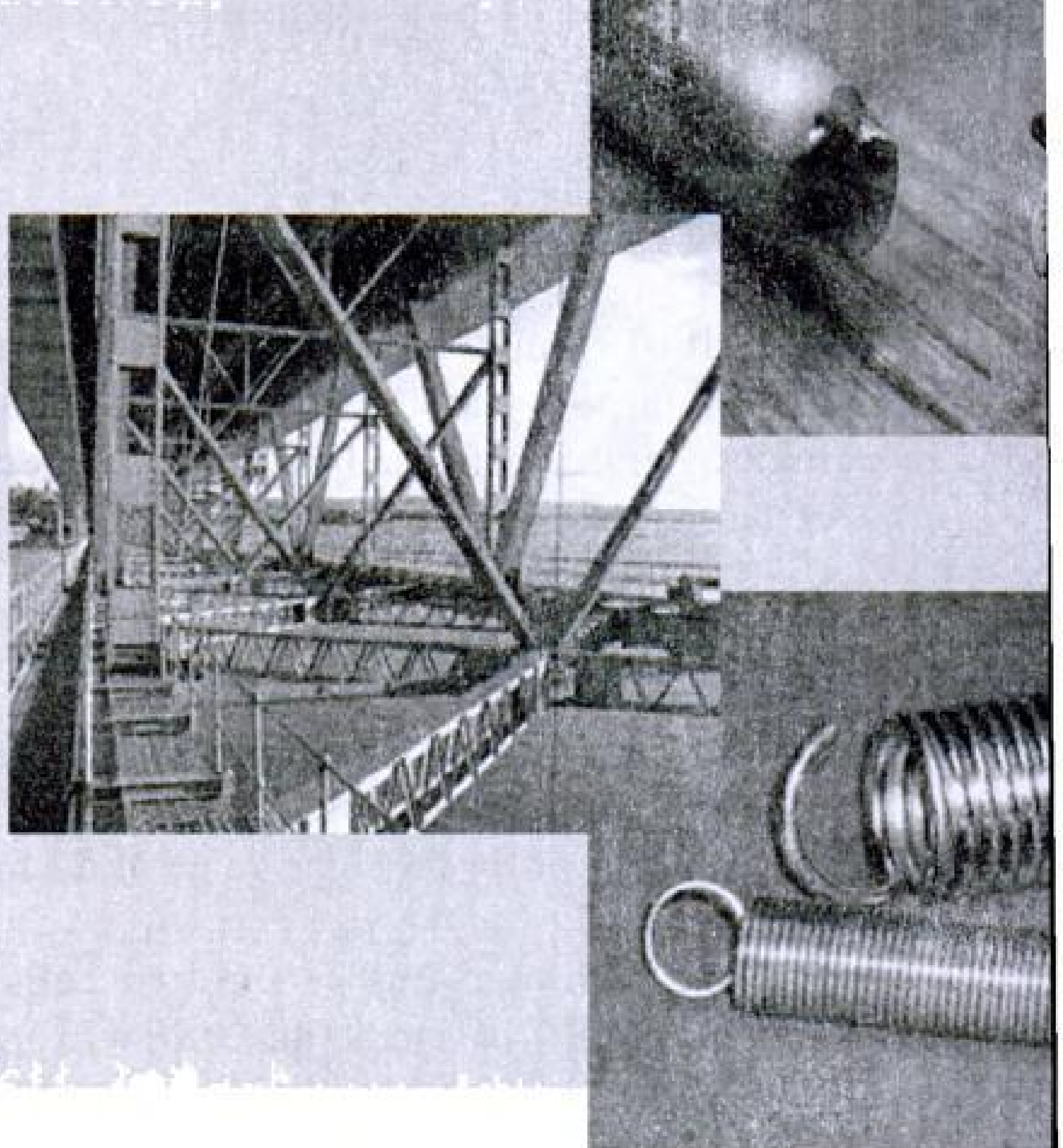


Fig. 10.14

11

SPRINGS



11.1 INTRODUCTION

Any elastic member which can deform under a force can act as a spring. The main function of a spring is to deflect under a load and to recover the original shape when the load is released. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses, e.g. a steel helical spring may be expanded to twice its length without loosing its elasticity. Springs can be made to act under tension, compression, torsion, bending or combination of these loads. Usually, the springs are made from conventional metals though sometimes they can be of non-metallic materials.

In general, the springs are used to

- absorb energy and to release the same according to the desired function to be performed such as a spring of a clock.
- absorb shocks as in case of automobiles.
- deflect under external forces to provide the desired motion to a machine member such as springs used in weighing machines, safety valves, clutches, governors, etc.

- *Stiffness* of a spring is defined as the force required for unit deflection.
- *Solid length* of a spring is the length of a spring in the fully compressed state when the coils touch each other.

11.2 CLOSE-COILED HELICAL SPRINGS

Close-coiled helical springs are those in which the angle of helix is so small that if the axis of the spring is vertical, the coils may be assumed to be in a horizontal plane. Such a spring may be acted upon by an axial load or an axial torque.

Solution

$$s = 600 \text{ N/m or } 0.6 \text{ N/mm}; W = 40 \text{ N}; \tau = 100 \text{ MPa}$$

Solid length, $nd = 60$ mm or $n = 60/d$

Relation between d and D can be obtained from the relation, $\tau = \dfrac{8WD}{\pi d^3}$

That is $$100 = \frac{8 \times 40 \times D}{\pi \times d^3} \text{ or } D = 0.982\, d^3$$

Stiffness s of the spring can be obtained from the relation, $\delta = \dfrac{8WD^3 n}{Gd^4}$

That is $$s = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n} \text{ or } 0.6 = \frac{32\,000 \times d^4}{8(0.982 d^3)^3 \times 60/d}$$

$$d^4 = 117.3 \text{ or } d = 3.29 \text{ mm}$$

$$\therefore \quad D = 0.982 \times 3.29^3 = 35 \text{ mm}$$

Number of coils, $n = 60/3.29 = 18.24$

Example 11.5 *A close-coiled helical spring made of 10-mm diameter steel bar has 8 coils of 150-mm mean diameter. Calculate the elongation, torsional stress and the strain energy per unit volume when the spring is subjected to an axial load of 130 N. Take G = 80 GPa.*

If instead of the axial load, an axial torque of 9 N.m is applied, find the axial twist, bending stress and the strain energy per unit volume. E = 205 GPa.

Solution

Under axial load, $$\delta = \frac{8WD^3 n}{Gd^4} = \frac{8 \times 130 \times 150^3 \times 8}{80\,000 \times 10^4} = 35.1 \text{ mm}$$

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 130 \times 150}{\pi \times 10^3} = 49.66 \text{ MPa}$$

Strain energy/unit volume = $\dfrac{\tau^2}{4G}$ ...(Eq. 10.5)

$$\frac{49.66^2}{4 \times 80\,000} = 0.007\,71 \text{ N.mm/mm}^3 \text{ or } 7710 \text{ N.m/m}^3 \text{ or } 7.71 \text{ kN.m/m}^3$$

Under axial torque, $$\varphi = \frac{64TDn}{Ed^4} = \frac{64 \times 9000 \times 150 \times 8}{205\,000 \times 10^4} = 0.3372 \text{ rad}$$

Bending stress, $$\frac{32T}{\pi d^3} = \frac{32 \times 9000}{\pi \times 10^3} = 91.7 \text{ MPa}$$

Strain energy/unit volume $\dfrac{T\varphi/2}{\text{Volume}}$

$$\frac{9000 \times 0.3372/2}{(\pi \times 10^2/4)\pi \times 150 \times 8} = 0.005\,124 \text{ N.mm/mm}^3 \text{ or } 5.124 \text{ kN.m/m}^3$$

$$= T\frac{\pi Dn}{\cos\alpha}\frac{32}{\pi d^4}\left(\frac{\sin^2\alpha}{G}+\frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32TDn}{d^4.\cos\alpha}\left(\frac{\sin^2\alpha}{G}+\frac{2\cos^2\alpha}{E}\right) \qquad (11.10)$$

$$\delta = \left(\frac{\partial U}{\partial T}\right)_{W=0} = \frac{2(T\sin\alpha)R\cos\alpha.l}{2GJ}+\frac{2(T\cos\alpha)(-R\sin\alpha)l}{2EI}$$

$$= TRl\sin\alpha\cos\alpha\left(\frac{1}{GJ}-\frac{1}{EI}\right)$$

$$= T\frac{D}{2}\frac{\pi Dn}{\cos\alpha}\frac{32}{\pi d^4}\sin\alpha\cos\alpha\left(\frac{1}{G}-\frac{2}{E}\right)$$

$$= \frac{16TD^2n\sin\alpha}{d^4}\left(\frac{1}{G}-\frac{2}{E}\right) \qquad (11.11)$$

Example 11.11 *An open-coiled helical spring has 12 turns wound to a mean diameter of 100 mm. The angle of the coils with a plane perpendicular to the axis of the coil is 30°. The wire diameter is 8 mm. Determine*

(i) the axial extension with a load of 80 N.

(ii) the angle turned by the free end if free to rotate.

E = 205 MPa and G = 80 GPa.

Solution

$$\delta = \frac{8WD^3n}{d^4.\cos\alpha}\left(\frac{\cos^2\alpha}{G}+\frac{2\sin^2\alpha}{E}\right) = \frac{8\times 80\times 100^3\times 12}{8^4.\cos 30°}\left(\frac{\cos^2 30°}{80\ 000}+\frac{2\sin^2 30°}{205\ 000}\right)$$

$$= 2.165\times 10^6(9.375\times 10^{-6}+2.439\times 10^{-6}) = 25.88 \text{ mm}$$

$$\varphi = \frac{16TD^2n\sin\alpha}{d^4}\left(\frac{1}{G}-\frac{2}{E}\right) = \frac{16\times 80\times 100^2\times 12\sin 30°}{8^4}\left(\frac{1}{80\ 000}-\frac{2}{205\ 000}\right)$$

$$= 18\ 750\times 2.744\times 10^{-6} = 0.0515 \text{ rad} \quad \text{or} \quad 0.0515\times\frac{180}{\pi} = 2.95°$$

Example 11.12 *An open-coiled helical spring has 10 coils and is made out of a 12-mm diameter steel rod. The mean diameter of the coils is 80 mm and the helix angle 15°. Find the deflection under an axial load of 250 N. What are the maximum intensities of direct and shear stresses induced in the section of the wire?*

If the above axial load is replaced by an axial torque of 60 N.m, determine the axial deflection and the angle of rotation about the axis of the coil. G = 80 MPa and E = 204 GPa.

$$U = \int \frac{T^2}{2EI}.ds = \frac{1}{2EI}\int (F_y.x - F_x.y)^2.ds \qquad \text{(i)}$$

As O is the fixed end of the spiral strip, deflection in x-direction is zero. Applying Castigliano's theorem,

$$\frac{\partial U}{\partial R_x} = 0 \text{ or } \int \frac{(F_y.x - F_x.y)(-y)}{EI}.ds = 0$$

or $\int (F_y.x - F_x.y)(-y)ds = 0$ or $F_x\int y^2.ds - F_y\int xy.ds = 0$

or
$$F_x = \frac{F_y\int xy.ds}{\int y^2.ds}$$

Due to symmetry of the spiral curve about the x-axis, $F_y\int xy.ds = 0$

$\therefore \qquad F_x = 0$

Thus
$$U = \frac{1}{2EI}\int (F_y.x)^2.ds \quad U = \int \frac{(T.x/R)^2}{2EI}.ds \qquad (\text{as } T = F_y.R)$$

Equating the strain energy to work done,

$$\frac{1}{2}T.\theta = \int \frac{(T.x/R)^2}{2EI}.ds \text{ where } \theta \text{ is the rotation of the spiral}$$

$$\theta = \frac{T}{EIR^2}\int x^2.ds$$

The integral $\int x^2.ds$ is found to be approximately equal to $1.25R^2l$

Therefore,
$$\theta = \frac{1.25R^2lT}{EIR^2} = \frac{1.25Tl}{EI} \qquad (11.12)$$

Maximum bending moment = $F_y.2R = 2T$ at the left-hand edge of the spiral

If b is the width and t the thickness of the spiral material,

$$\text{Maximum stress } \sigma_{max} = \frac{2T}{Z} = \frac{2T}{bt^2/6} = \frac{12T}{bt^2} \qquad (11.13)$$

Example 11.16 *A 5-mm wide and 0.3-mm thick flat spiral spring is 2-m long. The maximum stress is 600 MPa at the point of greatest bending moment. Determine the torque, the work stored and the number of turns to wind up the spring. E = 208 GPa.*

Solution

Maximum stress, $\sigma_{max} = \frac{12T}{bt^2}$ or $600 = \frac{12T}{5\times 0.3^2}$ or $T = 22.5$ N

$$\theta = \frac{1.25Tl}{EI} = \frac{1.25\times 22.5\times 2000}{208\ 000\times(5\times 0.3^3/12)} = 24 \text{ rad}$$

$$n = \frac{1625}{5^3} = 13$$

The actual deflection for a proof load of 6 kN,

$$\delta = \frac{3}{8} \cdot \frac{6000 \times 650^3}{13 \times 50 \times 5^3 \times 205\ 000} = 37.1 \text{ mm}$$

At proof load, the spring is straight, therefore, the initial radius of curvature is

$$R = \frac{l^2}{8\delta} = \frac{650^2}{8 \times 37.1} = 1424 \text{ mm or } 1.424 \text{ m}$$

Example 11.20 *A quarter-elliptic type of laminated spring is 800-mm long. The static deflection of the spring under an end load of 3 kN is 100 mm. Determine the number of leaves required and the maximum stress if the leaf is 75-mm wide and 8-mm thick. Also find the height from which a load can be dropped on to the undeflected spring to induce a maximum stress of 750 MPa. E = 208 GPa.*

Solution

$$\delta = \frac{6Wl^3}{nbt^3E} \text{ or } 100 = \frac{6 \times 3000 \times 800^3}{n \times 75 \times 8^3 \times 208\ 000};\ n = 11.54 \text{ say 12 leaves}$$

$$\sigma = \frac{6Wl}{nbt^2} = \frac{6 \times 3000 \times 800}{12 \times 75 \times 8^2} = 250 \text{ N/mm}^2$$

Equivalent gradually applied load to induce a maximum stress of 750 MPa

$$= 3000 \times \frac{750}{250} = 9000 \text{ N}$$

The corresponding deflection, $\delta = \frac{6 \times 9000 \times 800^3}{12 \times 75 \times 8^3 \times 208\ 000} = 288.5$ mm

Loss of potential energy = Gain of strain energy

$$3000(h + 288.5) = \frac{1}{2} \times 9000 \times 288.5$$

Height from which the load can be dropped, $h = 432.8 - 288.5 = 144.3$ mm

Summary

- Springs are used to absorb energy and to release the same according to the desired function to be performed, or to absorb shocks or to deflect under external forces to provide the desired motion to a machine member.
- Any member which can deform under a force can act as a spring. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses.

Both Ends Hinged

Consider a strut initially straight acted upon by an axial load P through the centroid. The deflected shape of the strut is shown in Fig. 12.1. The X-axis is taken through the centroids of the end sections and the origin is assumed at O. At a section at a distance x from O, let y be the deflection from the central line.

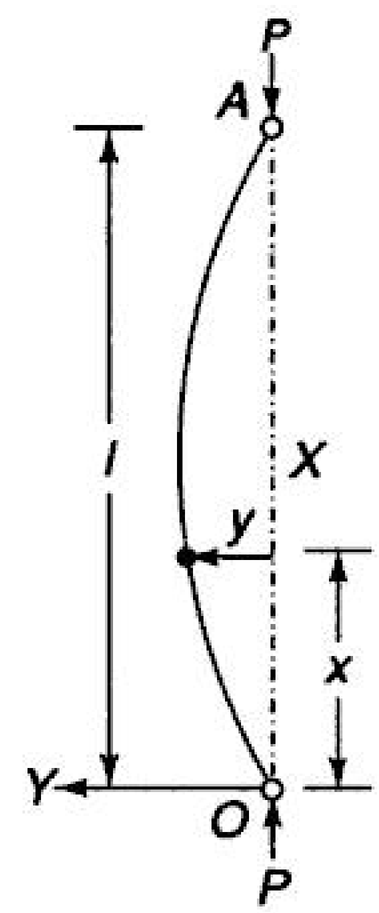


Fig. 12.1

From equation of bending of beams, $EI\dfrac{d^2y}{dx^2} = M = -Py$

To write the above equation with the proper sign is important. As the left-hand side indicates the bending moment from a positive value of deflection y, the Y-axis is to be taken in such a way that the deflection is positive, i.e. towards left in this case. Then viewing from the right side of the figure so that y-axis is upwards. Treating it as a beam, consider the bending moment at the point. As P provides counter-clockwise moment on the left of the section, it is negative. Had we taken deflection of the strut towards right, Y-axis would have been towards right and then viewing from left side of the figure P would provide clockwise moments on the right of section and thus negative.

The equation can be written as $\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \dfrac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x$

At $x = 0$, $y = 0$, $\therefore$ $B = 0$ and at $x = l$, $y = 0$ and thus $A\sin\alpha l = 0$

If $A = 0$, y is zero for all values of load and there is no bending.

$\therefore$ $\sin\alpha l = 0$ or $\alpha l = \pi$ (considering the least value)

or $\alpha = \pi/l$

$$\therefore \text{ Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2} \tag{12.1}$$

The higher solutions of $\sin\alpha l = 0$ leads to higher harmonics of the deflected column and practically are not important.

One End Fixed, Other Free (Fig. 12.2)

Take Y-axis towards right for positive value of y. Viewing from the left end, P provides a clockwise bending moment $P(a - y)$ on the left side and thus positive.

Then $EI\dfrac{d^2y}{dx^2} = M = P(a - y) = Pa - Py$

or $\dfrac{d^2y}{dx^2} + \alpha^2 y = \dfrac{P.a}{EI}$ where $\alpha^2 = \dfrac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{P.a}{EI\alpha^2}$

$= A\sin\alpha x + B\cos\alpha x + a$

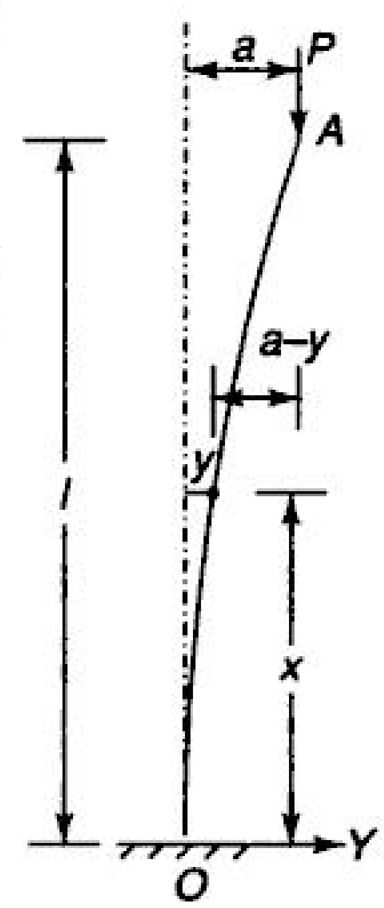


Fig. 12.2

$$\tau = \frac{\pi^2 I}{0.75\alpha l^2 A}\left(1 + \frac{AE}{sl}\right)$$ *where α is the coefficient of linear thermal expansion.*

Solution The average temp along the bar = $\frac{0.5t + t}{2} = 0.75t$

Thermal expansion of the bar = $0.75\alpha lt$

Let P be the force exerted by the spring on the bar on heating of the bar.

Compression of the bar under force $P = \frac{Pl}{AE}$

Net expansion of bar = Compression of spring

or $$0.75\alpha lt - \frac{Pl}{AE} = \frac{P}{s} \text{ or } 0.75\alpha lt = P\left(\frac{l}{AE} + \frac{1}{s}\right)$$

Thus $$P = \frac{0.75\alpha lt}{(l/AE + 1/s)}$$

and stress in the bar, $\sigma = \frac{0.75\alpha lt}{(l/E + A/s)}$

The bar will buckle when the load reaches to Euler's value i.e.

$$\frac{\pi^2 EI}{l^2} = \frac{0.75\alpha lt}{(l/AE + 1/s)} \quad \text{or} \quad \pi^2 EI\left(\frac{l}{AE} + \frac{1}{s}\right) = 0.75\alpha l^3 t$$

or $$t = \frac{\pi^2 EI}{0.75\,\alpha l^3}\left(\frac{l}{AE} + \frac{1}{s}\right) = \frac{\pi^2 I}{0.75\alpha l^2 A}\left(1 + \frac{EA}{sl}\right)$$

Example 12.6 *A straight cylindrical bar of 15-mm diameter and 1.2-m long is freely supported at its two ends in a horizontal position. It is loaded with a concentrated load of 100 N at the centre when the centre deflection is observed to be 5 mm. If placed in the vertical position and loaded vertically, what load would cause it to buckle? Also find the ratio of the maximum stress in the two cases.*

Solution Deflection due to centrmal load, $\delta = \frac{Wl^3}{48EI}$ or $\frac{EI}{l^2} = \frac{Wl}{48\,\delta}$

$\therefore$ Euler load, $P_e = \frac{\pi^2 EI}{l^2} = \pi^2 \frac{Wl}{48\,\delta} = \pi^2 \frac{100 \times 1200}{48 \times 5} = 4935$ N

As a strut maximum stress, $\sigma_e = \frac{P_e}{A} = \frac{4935}{\pi \times 15^2/4} = 27.93$ MPa

As a beam, maximum bending moment = $\frac{Wl}{4} = \frac{100 \times 1200}{4} = 30\ 000$ N.mm

Prof. Perry found that $\sec\dfrac{\pi}{2}\sqrt{\dfrac{P}{P_e}}=\dfrac{1.2P_e}{P_e-P}$ and thus $\sec\dfrac{\alpha l}{2}=\dfrac{1.2P_e}{P_e-P}$

Inserting this in the Secant formula of Eq. 12.14, $\sigma_{max}=\dfrac{P}{A}\left(1+\dfrac{e.y_c}{k^2}\cdot\dfrac{1.2P_e}{P_e-P}\right)$

Let $\quad \sigma_o=\dfrac{P}{A}$ and $\sigma_e=\dfrac{P_e}{A}$; Then $\sigma_{max}=\sigma_o\left(1+\dfrac{e.y_c}{k^2}\cdot\dfrac{1.2\sigma_e}{\sigma_e-\sigma_o}\right)$

$$\frac{\sigma_{max}}{\sigma_o}=-1=\frac{e.y_c}{k^2}\cdot\frac{1.2\sigma_e}{\sigma_e-\sigma_o}\text{ or }\left(\frac{\sigma_{max}}{\sigma_o}-1\right)\left(1-\frac{\sigma_o}{\sigma_e}\right)=\frac{1.2e.y_c}{k^2}\qquad(12.15)$$

This is Perry's approximate formula from which σ_o and hence P can easily be calculated.

Example 12.9 *An initially straight tube of 48-mm external diameter and 40-mm internal diameter in 2.4-m long and has hinged ends. It carries a compressive load of 25 kN parallel to the axis at an eccentricity of 2 mm. Determine the maximum and the minimum intensities of stresses in the tube. Also find the maximum permissible eccentricity so that no tension exists anywhere in the section. E =205 GPa.*

Solution

$$I=\frac{\pi}{64}\left(48^4-40^4\right)=42\,944\,\pi\text{ mm}^4;\ A=\frac{\pi}{4}\left(48^2-40^2\right)=176\pi\ \text{mm}^2$$

$$k^2=\frac{I}{A}=\frac{42\,944}{176}=244\ \text{mm}^2$$

$$\sec\frac{\alpha l}{2}=\sec\frac{l}{2}\sqrt{\frac{P}{EI}}=\sec\frac{2400}{2}\sqrt{\frac{25\,000}{205\,000\times 42\,944\pi}}=\sec 1.141=\sec 65.4°=2.4$$

$$\sigma_{max}=\frac{P}{A}\left(1+\frac{e.y_c}{k^2}.\sec\frac{\alpha l}{2}\right)=\frac{25\,000}{176\pi}\left(1+\frac{2\times 24}{244}\times 2.4\right)\text{MPa}$$

$$=45.22(1+0.472)=66.56\text{ MPa}\qquad\text{(compressive)}$$

$$\sigma_{min}=\frac{P}{A}\left(1+\frac{e.(-y_c)}{k^2}.\sec\frac{\alpha l}{2}\right)$$

$$=45.22(1-0.472)=23.88\text{ MPa}\qquad\text{(compressive)}$$

For maximum permissible eccentricity so that no tension exists anywhere in the section

$$\sigma_{min}=\frac{25\,000}{176\pi}\left(1+\frac{e\times(-24)}{244}\times 2.4\right)=0\ \text{ or }\ 1-0.236e=0\ \text{ or }e=4.24\text{ mm}$$

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$$= \frac{uE}{1-\nu^2}\left[\begin{array}{l} -\frac{P}{4\pi C}(\log x + 1) + \frac{P}{8\pi C} + \frac{1}{2}\frac{P}{4\pi C}\left(2\log R + \frac{1-\nu}{1+\nu}\right) \\ -\frac{P\nu}{4\pi C}\log x + \frac{P\nu}{8\pi C} + \frac{\nu}{2}\frac{P}{4\pi C}\left(2\log R + \frac{1-\nu}{1+\nu}\right) \end{array}\right]$$

$$= \frac{uE}{1-\nu^2}\left[\frac{P}{4\pi C}(\log R - \log x + \nu\log R - \nu\log x) + \frac{P}{8\pi C}\left(-2 + 1 + \frac{1-\nu}{1+\nu} + \nu + \nu.\frac{1-\nu}{1+\nu}\right)\right]$$

$$= \frac{uE}{1-\nu^2}\left[\frac{2P}{8\pi C}\left(\log\frac{R}{x}.(1+\nu)\right) + \frac{P}{8\pi C}\left\{-1 + \frac{1-\nu}{1+\nu} + \nu\left(1 + \frac{1-\nu}{1+\nu}\right)\right\}\right]$$

$$= \frac{uE}{1-\nu^2}\left[\frac{2P}{8\pi C}\left(\log\frac{R}{x}.(1+\nu)\right) + \frac{P}{8\pi C}\left\{\frac{-2\nu}{1+\nu} + \frac{2\nu}{1+\nu}\right\}\right]$$

$$= \frac{uE}{1-\nu^2}.\frac{2P}{8\pi C}.(1+\nu).\log\frac{R}{x}$$

$$= \frac{uE}{1-\nu^2}.\frac{2P}{8\pi}.(1+\nu).\log\frac{R}{x}\Big/\frac{Et^3}{12(1-\nu^2)}$$

$$= \frac{3}{2}.\frac{P}{\pi t^2}.(1+\nu).\log\frac{R}{x} \qquad (15.24)$$

Similarly, $\sigma_z = \frac{3}{2}.\frac{P}{\pi t^2}\left[(1+\nu).\log\frac{R}{x} + (1-\nu)\right]$ (15.25)

At the centre, these stresses become infinite theoretically. However, as the load cannot be a point load in the true sense, i.e. it must extend over a finite area, the maximum stresses will be finite in practice.

(ii) Edge clamped A plate firmly clamped at the edges and having a central point load is shown in Fig. 15.11.

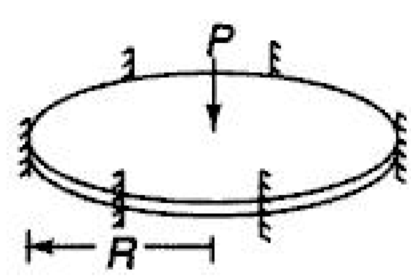

Fig. 15.11

As $w = 0$; ∴ $\theta = -\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1.\frac{x}{2} + \frac{C_2}{x}$

θ cannot be infinite at the centre,

∴ $C_2 = 0$ or $\theta = -\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1.\frac{x}{2}$

Now $y = -\frac{Px^2}{8\pi C}(\log x - 1) + C_1.\frac{x^2}{4} + C_3$

At $x = 0, y = 0$, ∴ $C_3 = 0$

∴ $y = -\frac{Px^2}{8\pi C}(\log R - 1) + C_1.\frac{x^2}{4}$

At $x = R$, $dy/dx = \theta = 0$, $0 = -\frac{PR}{4\pi C}\log R + \frac{PR}{8\pi C} + C_1.\frac{R}{2}$

$$-\frac{P}{8\pi C}\left[(1+\nu)2\log r+1-\nu\right]+\frac{C_1'}{2}(1+\nu)-\frac{C_2'}{r^2}(1-\nu)=\frac{C_1}{2}(1+\nu) \qquad \text{(iii)}$$

Also at $x = R$, $M_{xy} = 0$,

$$-\frac{P}{8\pi C}\left[(1+\nu)2\log R+1-\nu\right]+\frac{C_1'}{2}(1+\nu)-\frac{C_2'}{R^2}(1-\nu)=0 \qquad \text{(iv)}$$

- Multiplying (i) by $(1 + \nu)$

$$-\frac{P.r}{4\pi C}(1+\nu)\log r+\frac{P.r}{8\pi C}(1+\nu)+C_1'.\frac{r}{2}(1+\nu)+\frac{C_2'}{r}(1+\nu)=C_1.\frac{r}{2}(1+\nu) \qquad \text{(v)}$$

Multiplying (iii) by r,

$$-\frac{Pr}{8\pi C}\left[(1+\nu)2\log r+1-\nu\right]+\frac{C_1' r}{2}(1+\nu)-\frac{C_2'}{r}(1-\nu)=\frac{C_1 r}{2}(1+\nu) \qquad \text{(vi)}$$

Subtracting (vi) from (v),

$$-\frac{P.r}{8\pi C}-\frac{C_2'}{r}=0 \text{ or } C_2' = -\frac{P.r^2}{8\pi C}$$

- Multiplying (i) by

$$-\frac{P.r^2}{8\pi C}\log r+\frac{P.r^2}{16\pi C}+C_1'.\frac{r^2}{4}+\frac{C_2'}{2}=C_1.\frac{r^2}{4}$$

and equation (ii) is, $-\frac{P.r^2}{8\pi C}(\log r-1)+C_1'.\frac{r^2}{4}+C_2'.\log r+C_3' = C_1.\frac{r^2}{4}$

Subtracting first from the second,

$$C_3'+\frac{P.r^2}{16\pi C}-\frac{P.r^2}{8\pi C}\log r+\frac{P.r^2}{16\pi C}=0 \text{ or } C_3' = \frac{P.r^2}{8\pi C}(\log r-1)$$

- Equation (iv) is $-\frac{P}{8\pi C}\left[(1+\nu)2\log R+1-\nu\right]+\frac{C_1'}{2}(1+\nu)-\frac{C_2'}{R^2}(1-\nu)=0$

or $$-\frac{P}{8\pi C}\left[(1+\nu)2\log R+1-\nu\right]+\frac{C_1'}{2}(1+\nu)+\frac{P.r^2}{8\pi CR^2}(1-\nu)=0$$

or $$-\frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}\right]+C_1'+\frac{P.r^2}{4\pi CR^2}\frac{(1-\nu)}{(1+\nu)}=0$$

$$C_1' = \frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}\left(1-\frac{r^2}{R^2}\right)\right]$$

or $$C_1' = \frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}\left(\frac{R^2-r^2}{R^2}\right)\right]$$

Thus deflection is given by,

$$y = -\frac{Px^2}{8\pi C}(\log x-1)+C_1'.\frac{x^2}{4}+C_2'.\log x+C_3'$$

Summary

- In circular plates, freely supported at the edges and with uniformly distributed load,

  Deflection at the centre $= \dfrac{3wR^4}{16Et^3}(5+v)(1-v)$

  Radial and tangential stress at the centre $= \dfrac{3wR^2}{8t^2}(3+v)$ (maximum)

- In circular plates having clamped edges, and with uniformly distributed load,

  Deflection at the centre $= \dfrac{3wR^4}{16Et^3}(1-v^2)$

  Maximum radial stress $= \dfrac{3}{4}\dfrac{wR^2}{t^2}$ at clamped edge

  Maximum tangential stress $= \dfrac{3wR^2}{8t^2}(1+v)$ at centre

- In circular plates, freely supported at the edges and with a point load at the centre,

  Deflection at the centre $= \dfrac{3PR^2}{4\pi Et^3}.(3+v)(1-v)$

- In circular plates, freely supported at the edges and with a point load at the centre, the radial and tangential stresses become infinite at the centre theoretically. However, as the load cannot be a point load in the true sense, i.e. it must extend over a finite area, the maximum stresses will be finite in practice.
- In circular plates having clamped edges and with central point load,

  Deflection at the centre $= \dfrac{3}{4}.\dfrac{PR^2}{\pi Et^3}(1-v^2)$

  Maximum radial stress $= \dfrac{3}{2}.\dfrac{P}{\pi t^2}$ (at the edges)

  Maximum tangential stress $= \dfrac{3}{2}.\dfrac{Pv}{\pi t^2}$ (at the edges)

- In circular plates, freely supported at edges and with load round a circle of radius *r*,

  Central deflection, $y = \dfrac{3P(1-v^2)}{2\pi Et^3}\left[(R^2-r^2)\dfrac{(3+v)}{2(1+v)} - r^2.\log\dfrac{R}{r}\right]$

  Maximum values of radial and tangential stresses

  $$= \frac{3P}{4\pi t^2}\left[(1+v)2\log\frac{R}{r} + (1-v)\left(1-\frac{r^2}{R^2}\right)\right] \text{ at } x = r$$

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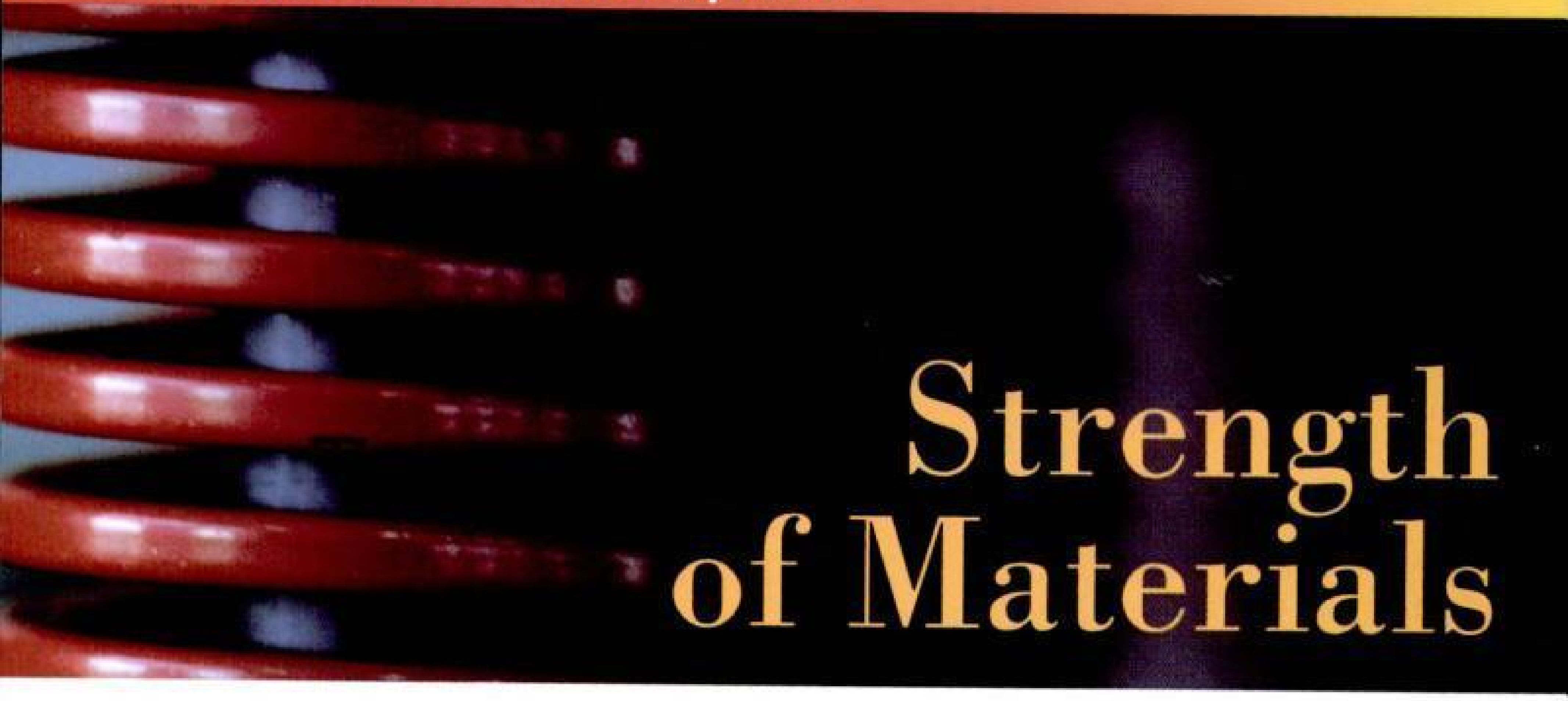

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