THE BOOK WAS DRENCHED

UNIVERSAL
LIBRARY

ANALYTICAL SOLID GEOMETRY

ANALYTICAL SOLID GEOMETRY

FOR

B. A. and B. Sc. (PASS and HONS.)

BY

SHANTI NARAYAN, M. A.,
Principal,
Hans Raj College, Delhi-6.
(*Delhi University*)

TWELFTH EDITION

Price : Rs. 6·00

1961

S. CHAND & CO.
DELHI—JULLUNDUR—LUCKNOW

Other works of the Author

Differential Calculus	Rs. 7·00
Integral Calculus	Rs. 6·25
Modern Pure Geometry	Rs. 5·00
Mathematical Analysis	Rs. 15·00
A Text Book of Matrices	Rs. 7·50
A Text Book of Vector Algebra	Rs. 6·50
A Text Book of Vector Calculus	Rs. 7·50
A Text Book of Cartesian Tensors	Rs. 6·50
Theory of Functions of a Complex Variable	Rs. 12·50
A Text Book of Modern Abstract Algebra	Rs. 14·00
A Text Book of General Topology (*Under preparation*)	

First	*Edition*	*1939*
Second	*Edition*	*1942*
Third	*Edition*	*1944*
Fourth	*Edition*	*1946*
Fifth	*Edition*	*1949*
Sixth	*Edition*	*1950*
Seventh	*Edition*	*1952*
Eighth	*Edition*	*1954*
Ninth	*Edition*	*1955*
Tenth	*Edition*	*1957*
Eleventh	*Edition*	*1959*
Twelfth	*Edition*	*1961*

S. CHAND & CO.

Asaf Ali Road —New Delhi
Fountain — Delhi
Mai Hiran Gate — Jullundur
Lal Bagh — Lucknow

Published by Shyam Lal Gupta, Managing Proprietor, S. Chand & Co., Delhi and Printed at R. K. Printers, 80-D, Kamla Nagar, Delhi-6,

PREFACE TO THE TWELFTH EDITION

A Chapter on General Equation of the second degree and reduction to canonical forms and classification has been added. It is hoped that the treatment is natural and simple and as such will appeal to the imagination of the students.

Hans Raj College, SHANTI NARAYAN
Delhi University,
January, 1961.

PREFACE TO THE FIRST EDITION

This book is intended as an introduction to Analytical Solid Geometry and covers as much of the subject as is generally expected of students going up for the B.A., B.Sc., Pass and Honours examinations of our Universities.

I have endeavoured to develop the subject in a systematic and logical manner. To help the beginner, elementary parts of the subject have been presented in as simple and lucid a manner as possible and fairly large number of solved examples to illustrate various types have been introduced. The books already existing in the market cover a rather extensive ground and consequently comparatively lesser attention is paid to the introductory portion than is necessary for a beginner.

The book contains numerous exercises of varied types in a graded form. Some of these have been selected from various examination papers and standard works to whose publishers and authors I offer my best thanks.

I am extremely indebted to Professor Sita Ram Gupta, M.A., P.E.S., of the Government College, Lahore, who very kindly went through the manuscript with great care and keen interest and suggested a large number of extremely valuable improvements.

I shall be very grateful for any suggestions for improvements or corrections of text or examples.

LAHORE : SHANTI NARAYAN
June, 1939.

CONTENTS

CHAPTER I

Co-ordinates

CHAPTER II

The Plane

CHAPTER III

Right Line

CHAPTER IV

Interpretation of Equations

Loci

CHAPTER V

Transformation of Co-ordinates

CHAPTER VI

The Sphere

CHAPTER VII

The Cone and Cylinder

APPENDIX

Homogeneous Cartesian Co-ordinates Elements At Infinity

CHAPTER VIII

The Conicoid

CHAPTER I

CO-ORDINATES

Introduction. In plane the position of a point is determined by two numbers x, y, obtained with reference to two straight lines in the plane generally at right angles. The position of a point in *space* is, however, determined by *three* numbers x, y, z. We now proceed to explain as to how this is done.

1·1. Co-ordinates of a point in space. Let $X'OX$, $Z'OZ$ be two perpendicular straight lines. Through O, their point of intersection,

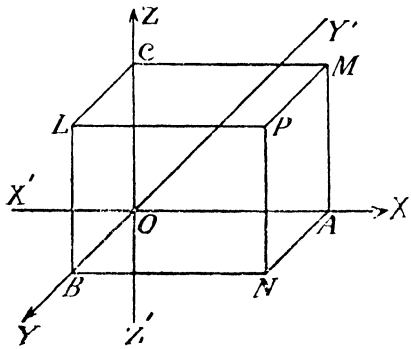

Fig. 1

called the *origin*, draw a line $Y'OY$ perpendicular to the XOZ plane so that we have three mutually perpendicular straight lines

$$X'OX,\ Y'OY,\ Z'OZ$$

known as *rectangular co-ordinate axes.* (The plane XOZ containing the lines $X'OX$ and $Z'OZ$ may be imagined as the plane of the paper ; the line OY as pointing towards the reader and OY' behind the paper). The positive directions of the axes are indicated by arrow heads. These three axes, taken in pairs, determine three planes,

$$XOY,\ YOZ \text{ and } ZOX$$

or briefly XY, YZ, ZX planes mutually at right angles, known as *rectangular co-ordinate planes.*

Through *any* point, P, in space, draw three planes parallel to the three co-ordinate planes (being also perpendicular to the corresponding axes) to meet the axes in A, B, C.

Let $OA=x$, $OB=y$ and $OC=z$.

These three numbers, x, y, z, determined by the point P, are called the *co-ordinates* of P.

Any one of these x, y, z, will be positive or negative according as it is measured from O, along the corresponding axis, in the positive or negative direction.

Conversely, given three numbers, x, y, z, we can find a point whose co-ordinates are x, y, z. To do this, we proceed as follows :

(*i*) Measure OA, OB, OC, along OX, OY, OZ equal to x, y, z respectively.

(*ii*) Draw through A, B, C planes parallel to the co-ordinate planes YZ, ZX and XY respectively.

The point where these three planes intersect is the required point P.

Note. The three co-ordinate planes divide the whole space in eight compartments which are known as *eight octants* and since each of the co-ordinates of a point may be positive or negative, there are $2^3(=8)$ points whose co-ordinates have the same numerical values and which lie in the eight octants, one in each.

1·11. Further explanation about co-ordinates. In § 1·1 above, we have learnt that in order to obtain the co-ordinates of a point P, we have to draw three planes through P respectively parallel to the three co-ordinate planes. The three planes through P and the three co-ordinate planes determine a parallelopiped whose consideration leads to three other useful constructions for determining the co-ordinates of P.

The parallelopiped, in question, has six rectangular faces

$$PMAN,\ LCOB\ ;\ PNBL,\ MAOC\ ;\ PLCM,\ NBOA$$

(*See Fig. 1*).

(*i*) We have

$x = OA = CM = LP$ = perpendicular from P on the YZ plane ;

$y = OB = AN = MP$ = perpendicular from P on the ZX plane ;

$z = OC = AM = NP$ = perpendicular from P on the XY plane.

Thus the co-ordinates x, y, z of any point P, are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

(*ii*) Since PA lies in the plane $PMAN$ which is perpendicular to the line OA*, therefore

$$PA \perp OA.$$

Similarly $PB \perp OB$ and $PC \perp OC$.

Thus the co-ordinates x, y, z of any point P are also the distances from the origin O of the feet A, B, C of the perpendiculars from the point to the co-ordinate axes $X'X$, $Y'Y$ and $Z'Z$ respectively.

---

* A line perpendicular to a plane is perpendicular to every line in the plane.

Ex. What are the perpendicular distances of a point (x, y, z) from the co-ordinate axes ? [*Ans.* $\sqrt{(y^2+z^2)}$, $\sqrt{(z^2+x^2)}$, $\sqrt{(x^2+y^2)}$

(*iii*) We have

$NP=AM=OC=z$;

$AN=OB=y$;

$OA=x$.

Thus (Fig. 2) if we draw $PN \perp XY$ *plane meeting it at* N *.and* $NA \parallel OY$ *meeting* OX *at* A, *we have*

$OA=x$, $AN=y$, $NP=z$.

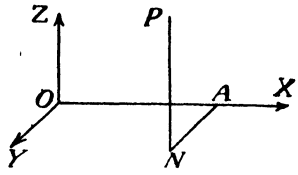

Fig. 2

Exercises

1. In fig. 1, write down the co-ordinates of A, B, C ; L, M, N when the co-ordinates of P are (x, y, z).

2. Show that for every point (x, y, z) on the ZX plane, $y=0$.

3. Show that for every point (x, y, z) on the Y-axis, $x=0$, $z=0$.

4. *What is the locus of a point for which*

(*i*) $x=0$, (*ii*) $y=0$, (*iii*) $z=0$.

(*iv*) $x=a$, (*v*) $y=b$, (*vi*) $z=c$.

5. *What is the locus of a point for which*

(*i*) $y=0$, $z=0$, (*ii*) $z=0$, $x=0$, (*iii*) $x=0$, $y=0$.

(*iv*) $y=b$, $z=c$, (*v*) $z=c$, $x=a$, (*vi*) $x=a$, $y=b$.

6. *P is any point (x, y, z), and α, β, γ are the angles which OP makes with x-axis, y-axis and z-axis respectively ; show that*

$$\cos\alpha=x/r,\ \cos\beta=y/r,\ \cos\gamma=z/r,$$

where $r=OP$.

7. Find the lengths of the edges of the rectangular parallelopiped formed by planes drawn through the points (1, 2, 3) and (4, 7, 6) parallel to the co-ordinate planes. [*Ans.* 3, 5, 3.

1·2. Distance between two points. *To find the distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.*

Through the points P, Q draw planes parallel to the co-ordinate planes to form a rectangular parallelopiped whose one diagonal is PQ.

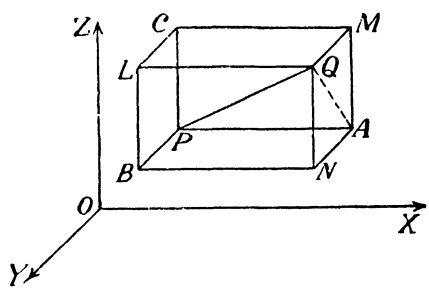

Fig. 3

Then

$APCM$, $NBLQ$; $LCPB$, $QMAN$; $BPAN$, $LCMQ$

are the three pairs of parallel faces of this parallelopiped,

Now $\angle ANQ$ is a rt. angle. Therefore,

$$AQ^2=AN^2+NQ^2.$$

Also AQ lies in the plane $QMAN$ which is perpendicular to the line PA. Therefore

$$AQ \perp PA.$$

Hence

$$PQ^2=PA^2+AQ^2=PA^2+AN^2+NQ^2.$$

Now, PA is the distance between the planes drawn through the points P and Q parallel to the YZ-plane and is, therefore, equal to the difference between their x—co-ordinates.

$\therefore$ $$PA=x_2-x_1.$$

Similarly $$AN=y_2-y_1,$$

and $$NQ=z_2-z_1.$$

Hence $$\mathbf{PQ^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.}$$

Thus the distance between the points

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2)$$

is $$\sqrt{[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2]}.$$

Cor. Distance from the origin. When P coincides with the origin O, we have $x_1=y_1=z_1=0$ so that we obtain,

$$OQ^2=x_2^2+y_2^2+z_2^2.$$

Note. The reader should notice the similarity of the formula obtained above for the distance between two points with the corresponding formula in plane co-ordinate geometry. Also refer §1·3.

Exercises

1. Find the distance between the points (4, 3, −6) and (−2, 1, −3). [*Ans.* **7.**

2. Show that the points (0, 7, 10), (−1, 6, 6), (−4, 9, 6) form an isosceles right-angled triangle.

3. Show that the three points (−2, 3, 5), (1, 2, 3), (7, 0, −1) are collinear.

4. Show that the points (3, 2, 2), (−1, 1, 3), (0, 5, 6), (2, 1, 2) lie on a sphere whose centre is (1, 3, 4). Find also its radius. [*Ans.* 3.

5. Find the co-ordinates of the point equidistant from the four points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$. [*Ans.* $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$.

1·3. Division of the join of two points. *To find the co-ordinates of the point dividing the line joining*

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2),$$

in the ratio $m : n$.

Let R (x, y, z) be the point dividing PQ in the ratio $m : n$.

Draw PL, QM, RN perpendiculars to the XY-plane.

The lines PL, QM, RN clearly lie in one plane so that the points L, M, N, lie in a straight line which is the intersection of this plane with the XY-plane.

The line through R parallel to the line LM shall lie in the same plane. Let it intersect PL and QM at H and K respectively.

The triangles HPR and QRK are similar so that we have

$$\frac{m}{n}=\frac{PR}{RQ}=\frac{PH}{KQ}=\frac{NR-LP}{MQ-NR}=\frac{z-z_1}{z_2-z}.$$

$$\therefore \quad z=\frac{mz_2+nz_1}{m+n}.$$

Similarly, by drawing perpendiculars to the XZ and YZ planes, we obtain

$$y=\frac{my_2+ny_1}{m+n} \text{ and } x=\frac{mx_2+nx_1}{m+n}.$$

The point R divides PQ internally or externally according as the ratio $m : n$ is positive or negative.

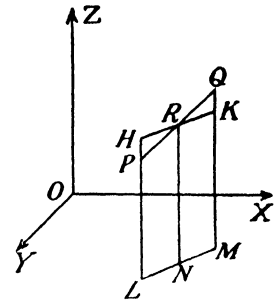

Fig. 4

Thus the co-ordinates of the point which divides the join of the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *in the ratio* $m : n$ *are*

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right).$$

Cor. 1. Co-ordinates of the middle point. In case R is the middle point of PQ, we have

$$m : n :: 1 : 1$$

so that

$$x=\tfrac{1}{2}(x_1+x_2),\ y=\tfrac{1}{2}(y_1+y_2),\ z=\tfrac{1}{2}(z_1+z_2).$$

Cor. 2. Co-ordinates of any point on the join of two points. Putting k for m/n, we see that the co-ordinates of the point R which divides PQ in the ratio $k : 1$ are

$$\left(\frac{kx_2+x_1}{1+k}, \frac{ky_2+y_1}{1+k}, \frac{kz_2+z_1}{1+k}\right).$$

To every value of k there corresponds a point R on the line PQ and to every point R on the line PQ corresponds some value of k, *viz.* PR/RQ.

Thus we see that the point

$$\left(\frac{kx_2+x_1}{1+k}, \frac{ky_2+y_1}{1+k}, \frac{kz_2+z_1}{1+k}\right) \qquad \ldots(i)$$

lies on the line PQ whatever value k may have and *conversely any* given point on the line PQ is obtained by giving *some* suitable value to k. This idea is sometimes expressed by saying that (i) are the *general co-ordinates* of any point of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Exercises

1. Find the co-ordinates of the points which divide the line joining the points $(2, -4, 3)$, $(-4, 5, -6)$ in the ratios

(i) $(1 : -4)$ and (ii) $(2 : 1)$.

[*Ans.* (i) $(4, -7, 6)$; (ii) $(-2, 2, -3)$.

2. A $(3, 2, 0)$, B $(5, 3, 2)$, C $(-9, 6, -3)$ are three points forming a triangle. AD, the bisector of the angle BAC, meets BC at D. Find the co-ordinates of D. [*Ans.* $(\frac{33}{16}, \frac{57}{16}, \frac{17}{16})$.

3. *Find the ratio in which the line joining the points*

$$(2, 4, 5), (3, 5, -4)$$

is divided by the YZ-plane.

The *general* co-ordinates of any point on the line joining the given points are

$$\left(\frac{3k+2}{1+k}, \frac{5k+4}{1+k}, \frac{-4k+5}{1+k}\right). \qquad \ldots(i)$$

This point will lie on the YZ plane, if, and only if, its x co-ordinate is zero, *i.e.*,

$$\frac{3k+2}{1+k}=0, \text{ i.e., } k=-\frac{2}{3}.$$

Hence the required ratio $=-2:3$. Putting $k=-2/3$ in (i), we see that the point of intersection is $(0, 2, 23)$.

4. Find the ratio in which the XY-plane divides the join of

$$(-3, 4, -8) \text{ and } (5, -6, 4).$$

Also obtain the point of intersection of the line with the plane.

[*Ans.* 2 ; $(7/3, -8/3, 0)$.

5. The three points $A(0, 0, 0)$, $B(2, -3, 3)$ $C(-2, 3, -3)$, are collinear. Find in what ratio each point divides the segment joining the other two.

[*Ans.* $AB/BC=-\frac{1}{2}$, $BC/CA=-2$, $CA/AB=1$.

6. Show that the following sets of points are collinear :

(*i*) $(2, 5, -4), (1, 4, -3), (4, 7, -6)$.

(*ii*) $(5, 4, 2), (6, 2, -1), (8, -2, -7)$.

7. Find the ratios in which the join of the points $(3, 2, 1)$, $(1, 3, 2)$ is divided by the locus of the equation

$$3x^2-72y^2+128z^2=3. \qquad [Ans. \quad -2:1 ; 1:-2.$$

8. $A(4, 8, 12)$, $B(2, 4, 6)$, $C(3, 5, 4)$, $D(5, 8, 5)$ are the four points ; show that the lines AB and CD intersect.

9. Show that the point $(1, -1, 2)$, is common to the lines which join $(6, -7, 0)$ to $(16, -19, -4)$ and $(0, 3, -6)$ to $(2, -5, 10)$.

10. Show that the co-ordinates of any point on the plane determined by the three points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), may be expressed in the form

$$\left(\frac{lx_1+mx_2+nx_3}{l+m+n}, \frac{ly_1+my_2+ny_3}{l+m+n}, \frac{lz_1+mz_2+nz_3}{l+m+n}\right).$$

11. Show that the centroid of the triangle whose vertices are (x_r, y_r, z_r) ; $r=1, 2, 3$, is

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

1·4. Tetrahedron. Tetrahedron is a figure bounded by four planes. It has four vertices, each vertex arising as a point of intersection of three of the four planes. It has six edges; each edge arising as the line of intersection of two of the four planes. $({}^4C_2=6)$.

To construct a tetrahedron, we start with three points A, B, C, and any point D, not lying on the plane determined by the points A, B, C. Then the four faces of the tetrahedron are the four triangles,

$$ABC, BCD, CAD, ABD;$$

the four vertices are the points

$$A, B, C, D$$

and the six edges are the lines

$$AB, CD ; BC, AD ; CA, BD.$$

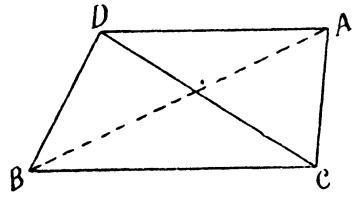

Fig. 5

The two edges AB, CD joining separately the points, A, B and C, D are called a pair of *opposite edges*. Similarly BC, AD and CA, BD are the two other pairs of opposite edges.

Exercises

1. The four lines drawn from the vertices of any tetrahedron to the centroids of the opposite faces meet in a point which is at three-fourths of the distance from each vertex to the opposite face.

2. Show that the three lines joining the mid-points of opposite edges of a tetrahedron meet in a point.

1·5. Angle between two lines. The meaning of the angle between two intersecting, *i.e.*, coplanar lines, is already known to the student. We now give the definition of the angle between two non-coplanar lines, also sometimes called *skew* lines.

Def. The angle between two *non-coplanar*, *i.e.*, non-intersecting lines is the angle between two *intersecting lines* drawn from *any* point parallel to each of the given lines.

Note 1. To justify the definition of angle between two non-coplanar lines, as given above, it is necessary to show that this angle is independent of the position of the point through which the parallel lines are drawn, but here we simply assume this result.

Note 2. The angles between a given line and the co-ordinate axes are the angles which the line drawn through the origin parallel to the given line makes with the axes.

1·6. Direction cosines of a line. If α, β, γ be the angles which any line makes with the positive directions of the axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction cosines* of the given line and are generally denoted by l, m, n respectively.

Ex. What are the direction cosines of the axes of co-ordinates ?

[*Ans.* 1, 0, 0 ; 0, 1, 0 ; 0, 0, 1.

1·61. A useful relation. *If O be the origin and (x, y, z) the co-ordinates of a point P, then*

$$x=lr,\ y=mr,\ z=nr,$$

where l, m, n are the direction cosines of OP and r, is the length of OP.

Through P draw $PL \perp x$-axis so that $OL=x$. From the rt. angled triangle OLP,

we have

$$\frac{OL}{OP}=\cos\angle LOP$$

$$i.e.,\ \frac{x}{r}=l \quad \text{or} \quad x=lr.$$

Similarly we have

$$y=mr,\ z=nr.$$

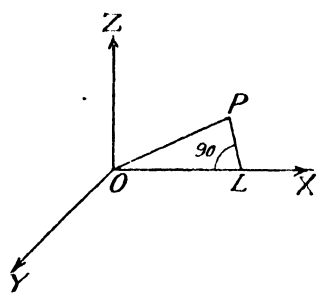

Fig. 6

1·7. Relation between direction cosines. *If l, m and n are the direction cosines of any line, then*

$$l^2+m^2+n^2=1,$$

i.e., the sum of the squares of the direction cosines of every line is one.

Let OP be drawn through the origin parallel to the given line so that l, m, n are the cosines of the angles which OP makes with OX, OY, OZ respectively. (*Refer Fig. 6*)

Let (x, y, z) be the co-ordinates of any point P on this line.

Let $$OP=r.$$

$\therefore$ $$x=lr,\ y=mr,\ z=nr.$$

Squaring and adding, we obtain

$$x^2+y^2+z^2=(l^2+m^2+n^2)r^2.$$

But $$x^2+y^2+z^2=OP^2=r^2.$$

$\therefore$ $$\mathbf{l^2+m^2+n^2=1.}$$

Cor. If a, b, c be three numbers *proportional* to the *actual* direction cosines l, m, n of a line, we have

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\pm\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{1}{\sqrt{(a^2+b^2+c^2)}}.$$

$\therefore$ $$l=\pm\frac{a}{\sqrt{(a^2+b^2+c^2)}},\ m=\pm\frac{b}{\sqrt{(a^2+b^2+c^2)}},$$

$$n=\pm\frac{c}{\sqrt{(a^2+b^2+c^2)}},$$

where the same sign, positive or negative, is to be chosen throughout.

Direction Ratios. From above, we see that a set of three numbers which are proportional to the actual direction cosines are sufficient to specify the direction of a line. Such numbers are called the **direction ratios.** Thus if a, b, c be the direction ratios of a line, its direction cosines are

$$\pm a/\sqrt{\Sigma a^2},\ \pm b/\sqrt{\Sigma a^2},\ \pm c/\Sigma a^2$$

Note. It is easy to see that if a line OP through the origin O makes angles α, β, γ with OX, OY, OZ, then the line OP obtained by producing OP

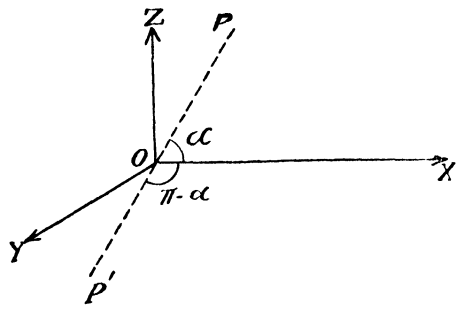

Fig. 7

backwards through O will make angles $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$ with OX, OY, OZ. Thus if

$$\cos\alpha=l,\ \cos\beta=m,\ \cos\gamma=n$$

are the direction cosines of OP, then

$$\cos(\pi-\alpha)=-l,\ \cos(\pi-\beta)=-m,\ \cos(\pi-\gamma)=-n$$

are the direction cosines of OP', *i.e.*, the line OP produced backwards.

Thus if we ignore the two senses of a line, we can think of the direction cosines l, m, n or $-l$, $-m$, $-n$ determining the direction of one and the same line. This explains the ambiguity in sign obtained above.

Note. The student should always make a distinction between direction cosines and direction ratios. It is only when l, m, n are direction cosines, that we have the relation

$$l^2+m^2+n^2=1.$$

Exercises

1. 6, 2, 3 are proportional to the direction cosines of a line. What are their actual values ? [*Ans.* (6/7, 2/7, 3/7).

2. What are the direction cosines of lines equally inclined to the axes ? How many such lines are there ? [*Ans.* $(1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$; 4.

3. The co-ordinates of a point P are (3, 12, 4). Find the direction cosines of the line OP. [*Ans.* (3/13, 12/13, 4/13).

4. *The direction cosines l, m, n, of two lines are connected by the relations*

$$l+m+n=0 \qquad \ldots(i)$$

$$2lm+2ln-mn=0. \qquad \ldots(ii)$$

Find them ?

Eliminating n between (i) and (ii) ; we get

$$2l^2-lm-m^2=0$$

or
$$2\left(\frac{l}{m}\right)^2-\frac{l}{m}-1=0. \qquad \ldots(iii)$$

This equation gives two values of l/m and hence there are two lines. The two roots of (iii) are 1 and $-\frac{1}{2}$.

If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two lines, we have

$$\frac{l_1}{m_1}=1 \text{ and } \frac{l_2}{m_2}=-\frac{1}{2}.$$

Also $\because\ l_1+m_1+n_1=0$ or $\frac{l_1}{m_1}+1+\frac{n_1}{m_1}=0$, $\therefore\ \frac{n_1}{m_1}=-2$,

and $\because\ l_2+m_2+n_2=0$ or $\frac{l_2}{m_2}+1+\frac{n_2}{m_2}=0$, $\therefore\ \frac{n_2}{m_2}=-\frac{1}{2}$.

$\therefore\ \frac{l_1}{1}=\frac{m_1}{1}=\frac{n_1}{-2}=\frac{1}{\sqrt{6}}$, or $l_1=\frac{1}{\sqrt{6}}$, $m_1=\frac{1}{\sqrt{6}}$, $n_1=-\frac{2}{\sqrt{6}}$.

and $\frac{l_2}{1}=\frac{m_2}{-2}=\frac{n_2}{1}=\frac{1}{\sqrt{6}}$, or $l_2=\frac{1}{\sqrt{6}}$, $m_2=-\frac{2}{\sqrt{6}}$, $n_2=\frac{1}{\sqrt{6}}$.

5. The direction cosines of two lines are determined by the relations

(i) $l-5m+3n=0$, $7l^2+5m^2-3n^2=0$;

(ii) $l+m-n=0$, $mn+6ln-12lm=0$;

find them ?

[*Ans.* (i) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$; $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

(ii) $\frac{1}{\sqrt{26}}, \frac{3}{\sqrt{26}}, \frac{4}{\sqrt{26}}$; $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

1·8. Projection on a Straight line.

1·81. Projection of a point on a line. The foot of the perpendicular P from a given point A on a given straight line BC is called the

orthogonal projection (or simply projection for the purpose of this book) of the point on the line and is the same point where the plane through the given point and perpendicular to the given line meets the line.

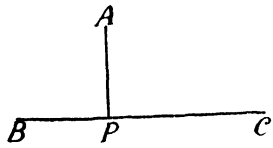

Fig. 8.

Thus in Fig. 1, page 1, A is the projection of P on X-axis ; also B and C are the projections of P on Y-axis and Z-axis respectively.

1·82. Projection of a segment of a line on another line. The projection of a segment AB of a line on any line CD is the segment $A'B'$ of CD where A', B' are the projections of A, B respectively on the line CD.

Clearly $A'B'$ is the intercept made on CD by planes through A, B each perpendicular to CD.

Ex. The co-ordinates of a point P are (x, y, z). What are the projections of OP on the co-ordinate axes ? [*Ans.* x, y, z.

Theorem. *The projection of a given segment AB of a line on any line CD is $AB \cos \theta$, where θ is the angle between AB and CD.*

Let the planes through A and B perpendicular to the line CD meet it in A', B' respectively so that $A'B'$ is the projection of AB.

Through A draw a line $AP \parallel CD$ to meet the plane through B at P.

Now, $AP \parallel CD$.

$\therefore$ $\angle PAB = \theta$.

Also BP lies in the plane which is $\perp AP$.

$\therefore$ $\angle APB = 90°$

Hence $AP = AB \cos \theta$

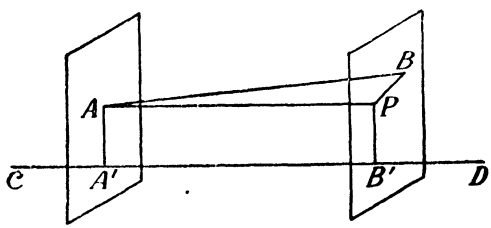

Fig. 9

Clearly $A'B'PA$ is a rectangle so that we have

$$AP = A'B'.$$

Hence $A'B' = AB \cos \theta$.

Cor. Direction cosines of the join of two points.

To find the direction cosines of the line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Let L, M be the feet of the perpendiculars drawn from P, Q to the X-axis respectively so that

$$OL=x_1,\ OM=x_2.$$

Projection of PQ on X-axis $=LM$

$$=OM-OL$$

$$=x_2-x_1.$$

Also if l, m, n be the direction cosines of PQ, the projection of PQ on X-axis $=l.PQ$.

$$\therefore \qquad l.PQ=x_2-x_1.$$

Similarly projecting PQ on Y-axis and Z-axis, we get

$$m.PQ=y_2-y_1,$$

$$n.PQ=z_2-z_1.$$

$$\therefore \qquad \frac{x_2-x_1}{l}=\frac{y_2-y_1}{m}=\frac{z_2-z_1}{n}=PQ.$$

Thus the direction cosines of the line joining the two points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are proportional to*

$$x_2-x_1,\ y_2-y_1,\ z_2-z_1.$$

Exercises

1. Find the direction cosines of the lines joining the points
   (*i*) (4, 3, −5) and (−2, 1, −8). [*Ans.* (6/7, 2/7, 3/7)
   (*ii*) (7, −5, 9) and (5, −3, 8). [*Ans.* (2/3, −2/3, 1/3)
2. Show that the points (1, −2, 3), (2, 3, −4), (0, −7, 10) are collinear.
3. The projections of a line on the axes are 12, 4, 3. Find the length and the direction cosines of the line. [*Ans.* 13 ; (12/13, 4/13, 3/13)

1·83. Projection of a broken line (*consisting of several continuous segments*). *If* $P_1, P_2, P_3, \ldots\ldots, P_n$ *be any number of points in space, then the sum of the projections of*

$$P_1P_2,\ P_2P_3, \ldots\ldots,\ P_{n-1}P_n$$

on any line is equal to the projection of P_1P_n *on the same line.*

Let $Q_1, Q_2, Q_3, \ldots\ldots, Q_n$

be the projections of the points $P_1, P_2, P_3, \ldots\ldots, P_n$

on the given line. Then

Q_1Q_2 = projection of P_1P_2,

Q_2Q_3 = ,, ,, P_2P_3,

and so on.

Also Q_1Q_n = projection of P_1P_n.

As $Q_1, Q_2, Q_3 \ldots\ldots, Q_n$ lie on the same line we have, for all relative positions of these points on the line, the relation

$$Q_1Q_2+Q_2Q_3+\ldots\ldots+Q_{n-1}Q_n=Q_1Q_n$$

Hence the theorem.

1·84. Projection of the join of two points on a line. *To show that the projection of the line joining*

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

on a line with direction cosines l, m, n is

$$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n.$$

Through P, Q draw planes parallel to the co-ordinate planes to form a rectangular parallelopiped whose one diagonal is PQ. (*See Fig. 3, Page 3*).

Now

$$PA=x_2-x_1,\ AN=y_2-y_1,\ NQ=z_2-z_1.$$

The lines PA, AN, NQ are respectively parallel to x-axis, y-axis, z-axis. Therefore, their respective projections on the line with direction cosines l, m, n are

$$(x_2-x_1)l,\ (y_2-y_1)m,\ (z_2-z_1)n.$$

As the projection of PQ on any line is equal to the sum of the projections of PA, AN, NQ on that line, therefore the required projection is

$$\mathbf{(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n.}$$

Exercises

1. $A(6, 3, 2)$, $B(5, 1, 4)$, $C(3, -4, 7)$, $D(0, 2, 5)$ are four points. Find the projections of AB on CD and of CD on AB. [*Ans.* $-13/7$; $-13/3$.

2. Show by projection that if P, Q, R, S are the points $(6, -6, 0)$, $(-1, -7, 6)$, $(3, -4, 4,)$ $(2, -9, 2)$ respectively then $PQ \perp RS$.

1·9. Angle between two lines. *To find the angle between lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2).*

Let OP_1, OP_2, be lines through the origin parallel to the given lines so that the cosines of the angles which OP_1 and OP_2 make with

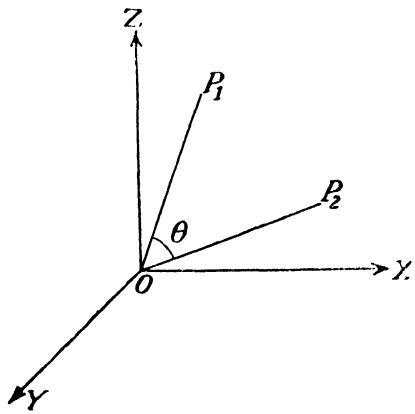

Fig. 10

the axes are l_1, m_1, n_1 and l_2, m_2, n_2, respectively and the angle between the given lines is the angle between OP_1 and OP_2. Let this angle be θ.

Let the co-ordinates of P_2 be (x_2, y_2, z_2).

The projection of the line OP_2 joining

$$O(0,\ 0,\ 0) \text{ and } P_2(x_2,\ y_2,\ z_2)$$

on the line OP_1 whose direction cosines are

$$l_1,\ m_1,\ n_1,$$

is $$(x_2-0)l_1+(y_2-0)m_1+(z_2-0)n_1=l_1x_2+m_1y_2+n_1z_2$$

Also this projection is $OP_2 \cos\theta$.

$$\therefore \quad OP_2 \cos\theta=l_1x_2+m_1y_2+n_1z_2.$$

But $$x_2=l_2.OP_2,\ y_2=m_2.OP_2,\ z_2=n_2.OP_2. \qquad (\S1{\cdot}61)$$

$$\therefore \quad OP_2 \cos\theta=(l_1l_2+m_1m_2+n_1n_2)OP_2$$

or $$\cos\theta=\mathbf{l_1l_2+m_1m_2+n_1n_2}.$$

Second Method. Suppose $OP_1=r_1$, $OP_2=r_2$.

Let the co-ordinates of P_1, P_2, be $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ respectively.

Then $$x_1=r_1l_1,\quad y_1=r_1m_1,\quad z_1=r_1n_1, \qquad (\S1{\cdot}61)$$

and $$x_2=r_2l_2,\quad y_2=r_2m_2,\quad z_2=r_2n_2.$$

We have

$$P_1P_2{}^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$$
$$=(x_2{}^2+y_2{}^2+z_2{}^2)+(x_1{}^2+y_1{}^2+z_1{}^2)-2(x_1x_2+y_1y_2+z_1z_2)$$
$$=r_2{}^2+r_1{}^2-2r_2r_1(l_1l_2+m_1m_2+n_1n_2). \qquad \ldots(i)$$

Also from Trigonometry, we have

$$P_1P_2{}^2=r_1{}^2+r_2{}^2-2r_1r_2 \cos\theta \qquad \ldots(ii)$$

Therefore, from (i) and (ii), we obtain

$$r_1{}^2+r_2{}^2-2r_1r_2 \cos\theta=P_1P_2{}^2$$
$$=r_1{}^2+r_2{}^2-2r_1r_2(l_1l_2+m_1m_2+n_1n_2)$$

i.e., $$\cos\theta=l_1l_2+m_1m_2+n_1n_2.$$

Cor. 1. Sin θ and tan θ. The expressions for sin θ and tan θ in a convenient form are obtained as follows :—

$$\sin^2\theta=1-\cos^2\theta$$
$$=1-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(l^2{}_1+m_1{}^2+n_1{}^2)(l_2{}^2+m_2{}^2+n_2{}^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$
$$=(l_1m_2-l_2m_1)^2+(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2.$$

$$\therefore \quad \sin\theta=\pm\sqrt{[\Sigma(l_1m_2-l_2m_1)^2]}$$

and $$\tan\theta=\frac{\sin\theta}{\cos\theta}=\pm\frac{\sqrt{[\Sigma(l_1m_2-l_2m_1)^2]}}{\Sigma l_1l_2}.$$

Cor. 2. If the direction cosines of two lines be proportional to a_1, b_1, c_1, and a_2, b_2, c_2, then their actual values are

$$\pm\frac{a_1}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2)}},\ \pm\frac{b_1}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2)}},\ \pm\frac{c_1}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2)}};$$

$$\pm\frac{a_2}{\sqrt{(a_2{}^2+b_2{}^2+c_2{}^2)}},\ \pm\frac{b_2}{\sqrt{(a_2{}^2+b_2{}^2+c_2{}^2)}},\ \pm\frac{c_2}{\sqrt{(a_2{}^2+b_2{}^2+c_2{}^2)}},$$

so that if θ be the angle between the given lines, we have

$$\cos\theta = \pm \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}},$$

$$\sin\theta = \pm \frac{\sqrt{[(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2]}}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}},$$

$$\tan\theta = \pm \frac{\sqrt{[\Sigma(a_1b_2-a_2b_1)^2]}}{\Sigma a_1a_2}.$$

The expression for $\tan\theta$ is of the same form whether we use direction cosines or direction ratios.

Cor. 3. Conditions for perpendicularity and parallelism.

(*i*) When the given lines are perpendicular,

$$\theta = 90^\circ \text{ so that } \cos\theta = 0.$$

This gives

$$\mathbf{a_1a_2+b_1b_2+c_1c_2=0}.$$

(*ii*) When the given lines are parallel,

$$\theta = 0 \text{ so that } \sin\theta = 0.$$

This gives

$$(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2=0,$$

which is true only when

$$a_1b_2-a_2b_1=0,\ b_1c_2-b_2c_1=0,\ c_1a_2-c_2a_1=0,$$

or

$$\frac{\mathbf{a_1}}{\mathbf{a_2}}=\frac{\mathbf{b_1}}{\mathbf{b_2}}=\frac{\mathbf{c_1}}{\mathbf{c_2}}.$$

This result is also otherwise evident, for, the lines through the origin drawn parallel to the parallel lines coincide and, therefore, their direction cosines must be the same and hence direction ratios proportional.

Exercises

1. Find the angles between the lines whose direction ratios are

(*i*) 5, −12, 13 ; −3, 4, 5. [*Ans.* $\cos^{-1}(1/65)$.

(*ii*) 1, 1, 2 ; $\sqrt{3}-1$, $-\sqrt{3}-1$, 4. [*Ans.* $\pi/3$.

2. Show that the angle between the lines whose direction cosines are given by the relations in Ex. 4, P. 9 is $\frac{1}{3}\pi$.

3. *Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to* (*1*, −*2*, −*2*), (*0*, *2*, *1*).

Sol. If l, m, n be the direction cosines of the line perpendicular to the given lines, we have

$$l.1+m(-2)+n(-2)=0, \textit{ i.e., } l-2m-2n=0,$$
$$l(0)+m(2)+n(1)=0, \textit{ i.e., } 0l+2m+n=0.$$

These give

$$\frac{l}{2}=\frac{m}{-1}=\frac{n}{2}.$$

$$\therefore\quad l=\frac{2}{\sqrt{[2^2+(-1)^2+2^2]}}=\frac{2}{3},\ m=-\frac{1}{3},\ n=\frac{2}{3}.$$

4. Show that a line can be found perpendicular to the three lines with direction cosines proportional to (2, 1, 5), (4, −2, 2), (−6, 4, −1). Hence show that if these three lines be concurrent, they are also coplanar.

5. *l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Show that the direction cosines of the line perpendicular to them both are*

$$m_1n_2-m_2n_1,\ n_1l_2-n_2l_1,\ l_1m_2-l_2m_1.$$

Sol. If l, m, n be the direction cosines of the required line, we have

$$ll_1+mm_1+nn_1=0,$$
$$ll_2+mm_2+nn_2=0.$$

These give

$$\frac{1}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}=\frac{\sqrt{\Sigma l^2}}{\sqrt{[\Sigma(m_1n_2-m_2n_1)^2]}}=\frac{1}{\sin\theta},$$

where θ is the angle between the given lines. As $\theta=90°$, we have $\sin\theta=1$. Hence the result.

6. l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of two intersecting lines. Show that lines through the intersection of these two with direction ratios

$$l_1+kl_2,\ m_1+km_2,\ n_1+kn_2$$

are coplanar with them ; k being any number whatsoever.

(Show that they all have a common perpendicular direction.)

7. Show that three concurrent lines with direction cosines

$$(l_1, m_1, n_1),\ (l_2, m_2, n_2),\ (l_3, m_3, n_3)$$

are coplanar if.

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=0.$$

8. Show that the join of points (1, 2, 3), (4, 5, 7) is parallel to the join of the points (−4, 3, −6), (2, 9, 2).

9. Show that the points

$$(4, 7, 8)\ ;\ (2, 3, 4)\ ;\ (-1, -2, 1)\ ;\ (1, 2, 5)$$

are the vertices of a parallelogram.

10. Show that the points

$$(5, -1, 1),\ (7, -4, 7),\ (1, -6, 10),\ (-1, -3, 4)$$

are the vertices of a rhombus.

11. Show that the points.

$$(0, 4, 1),\ (2, 3, -1),\ (4, 5, 0),\ (2, 6, 2)$$

are the vertices of a square.

12. $A(1, 8, 4)$, $B(0, -11, 4)$, $C(2, -3, 1)$ are three points and D is the foot of the perpendicular from A on BC. Find the co-ordinates of D.

[*Ans.* (4, 5, −2).

13. Find the point in which the join of (−9, 4, 5) and (11, 0, −1) is met by the perpendicular from the origin. [*Ans.* (1, 2, 2).

14. $A(-1, 2, -3)$, $B(5, 0, -6)$, $C(0, 4, -1)$ are three points. Show that the direction cosines of the bisectors of the angle BAC are proportional to (25, 8, 5) and (−11, 20, 23).

[*Hint.*—Find the co-ordinates of the points which divide BC in the ratio $AB : AC$.]

15. Find the angle between the lines whose direction cosines are given by the equations $3l+m+5n=0$ and $6mn-2nl+5lm=0$. [*Ans.* $\cos^{-1}\frac{1}{6}$.

16. Show that the pair of lines whose direction cosines are given by $3lm-4ln+mn=0$, $l+2m+3n=0$ are perpendicular.

17. *Show that the straight lines whose direction cosines are given by the equations*

$$al+bm+cn=0,\ ul^2+vm^2+wn^2=0$$

are perpendicular or parallel according as

$$a^2(v+w)+b^2(w+u)+c^2(u+v)=0 \quad \text{or} \quad a^2/u+b^2/v+c^2/w=0.$$

Sol. Eliminating l, between the given relations, we have

$$\frac{u(bm+cn)^2}{a^2}+vm^2+wn^2=0$$

or
$$(b^2u+a^2v)m^2+2umcn+(c^2u+a^2w)n^2=0 \qquad ...(i)$$

If the lines be parallel, their direction cosines are equal so that the two values of m/n must be equal. The condition for this is

$$u^2b^2c^2=(b^2u+a^2v)(c^2u+a^2w)$$

or

$$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0.$$

Again, if l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the two lines, then equation (i) gives

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{m_1m_2}{n_1n_2}=\frac{c^2u+a^2w}{b^2u+a^2v}$$

or

$$\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{b^2u+a^2v}$$

or

Similarly the elimination of n, gives, (or by symmetry)

$$\frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{a^2w+c^2u}$$

$$\therefore \qquad \frac{l_1l_2}{b^2w+c^2v}=\frac{m_1m_2}{a^2w+c^2u}=\frac{n_1n_2}{b^2u+a^2v}=k, \text{ say}$$

or

$$l_1l_2+m_1m_2+n_1n_2=k(b^2w+c^2v+a^2w+c^2u+b^2u+a^2v).$$

For perpendicular lines

$$l_1l_2+m_1m_2+n_1n_2=0$$

Thus the condition for perpendicularity is

$$a^2(v+w)+b^2(w+u)+c^2(u+v)=0.$$

18. Show that the straight lines whose direction cosines are given by

$$al+bm+cn=0,\ fmn+gnl+hlm=0,$$

are perpendicular if

$$f/a+g/b+h/c=0,$$

and parallel if

$$\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0.$$

19. *l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction cosines of two concurrent lines. Show that the direction cosines of the lines bisecting the angles between them are proportional to*

$$l_1\pm l_2,\ m_1\pm m_2,\ n_1\pm n_2.$$

Sol. Let the lines concur at the origin O and let OA, OB be the two lines. Take points A, B on the two lines such that $OA=OB=r$, say. Also take a

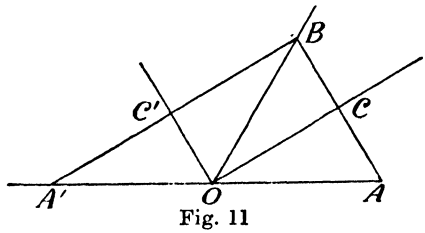

Fig. 11

point A' on AO produced such that $AO=OA'$. Let C, C' be the mid-points of AB and $A'B$. Then OC, OC' are the required bisectors. The result, now, follows from the fact that the co-ordinates of A, B, A' respectively are

$$(l_1r, m_1r, n_1r)\ ;\ (l_2r, m_2r, n_2r)\ ;\ (-l_1r, -m_1r, -n_1r),$$

20. *If the edges of a rectangular parallelopiped are a, b, c show that the angles between the four diagonals are given by*

$$\cos^{-1}[(\pm a^2 \pm b^2 \pm c^2)/(a^2+b^2+c^2)].$$

Sol. Take one of the vertices O of the parallelopiped as origin and the three rectangular faces through it as the three rectangular co-ordinate planes. (*See Fig. 1, Page 1*).

Let $OA=a$, $OB=b$, $OC=c$

The lines OP, AL, BM, CN are the four diagonals.

The co-ordinates of A, B, C are $(a, 0, 0)$; $(0, b, 0)$; $(0, 0, c)$.

" L, M, N are $(0, b, c)$; $(a, 0, c)$; $(a, b, 0)$.

" O, P are $(0, 0, 0)$; (a, b, c).

Direction cosines of OP are $\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$;

Direction cosines of AL are $\frac{-a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$;

" " " BM are $\frac{a}{\sqrt{\Sigma a^2}}, \frac{-b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$;

" " " CN are $\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{-c}{\sqrt{\Sigma a^2}}$.

The angle between OP and CN, therefore, is

$$\cos^{-1}\frac{a^2+b^2-c^2}{a^2+b^2+c^2}.$$

Similarly the angle between any one of the six pairs of diagonals can be found.

21. A line makes angles $\alpha, \beta, \gamma, \delta$, with the four diagonals of a cube ; prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$. (*P.U. 1932*)

(Choose axes as in Ex. 20 above and suppose that the direction cosines of the given line are l, m, n.)

22. *O, A, B, C, are four points not lying in the same plane and such that $OA \perp BC$ and $OB \perp CA$. Prove that $OC \perp AB$. What well-known theorem does this become if four points are co-planar ?*

The result of this example may also be stated thus :—

"If two pairs of opposite edges of a tetrahedron be at right angles, then so is the third."

Take O as origin and any three mutually perpendicular lines through O as co-ordinate axes.

Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the co-ordinates of the points A, B, C respectively.

As $OA \perp BC$, we have

$$x_1(x_2-x_3)+y_1(y_2-y_3)+z_1(z_2-z_3)=0, \qquad \ldots(i)$$

As $OB \perp CA$, we have

$$x_2(x_3-x_1)+y_2(y_3-y_1)+z_2(z_3-z_1)=0. \qquad \ldots(ii)$$

Adding (i) and (ii), we obtain

$$x_3(x_2-x_1)+y_3(y_2-y_1)+z_3(z_2-z_1)=0$$

which shows that $OC \perp AB$.

23. If, in a tetrahedron $OABC$,

$$OA^2+BC^2=OB^2+CA^2=OC^2+AB^2,$$

then its pairs of opposite edges are at right angles.

24. l_1, m_1, n_1 and l_2, m_2, n_2 are two directions inclined at an angle φ, to each other. Show that the direction

$$\frac{l_1+l_2}{2\cos\frac{1}{2}\varphi}, \frac{m_1+m_2}{2\cos\frac{1}{2}\varphi}, \frac{n_1+n_2}{2\cos\frac{1}{2}\varphi}$$

bisects the angle between these two directions.

Show that these direction cosines are the actual values.

25. Show that the direction equally inclined to the three mutually perpendicular directions

$$l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3,$$

is given by the direction cosines

$$\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}.$$

26. Show that the area of the triangle whose vertices are the origin and the points (x_1, y_1, z_1), and (x_2, y_2, z_2) is

$$\tfrac{1}{2}\sqrt{[(y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2+(x_1y_2-x_2y_1)^2]}. \qquad (B.U.\ 1958)$$

27. $l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3$

are the direction cosines of three mutually perpendicular lines ; show that

$$l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3$$

are also the direction cosines of three mutually perpendicular lines. Hence show that

$$\begin{aligned} l_1^2+m_1^2+n_1^2&=1, & l_1^2+l_2^2+l_3^2&=1, \\ l_2^2+m_2^2+n_2^2&=1, & m_1^2+m_2^2+m_3^2&=1, \\ l_3^2+m_3^2+n_3^2&=1, & n_1^2+n_2^2+n_3^2&=1, \\ l_1l_2+m_1m_2+n_1n_2&=0, & l_1m_1+l_2m_2+l_3m_3&=0, \\ l_2l_3+m_2m_3+n_2n_3&=0, & m_1n_1+m_2n_2+m_3n_3&=0, \\ l_3l_1+m_3m_1+n_3n_1&=0, & n_1l_1+n_2l_2+n_3l_3&=0. \end{aligned}$$

CHAPTER II

THE PLANE

2·1. General equation of first degree. *Every equation of the first degree in x, y, z represents a plane.*

The most *general* equation of the first degree in x, y, z *is*

$$ax+by+cz+d=0$$

where a, b, c are not all zero.

The locus of this equation will be a plane if *every* point of the line joining *any* two points on the locus also lies on the locus.

To show this, we take any two points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

on the locus, so that we have

$$ax_1+by_1+cz_1+d=0, \qquad \ldots(i)$$

$$ax_2+by_2+cz_2+d=0. \qquad \ldots(ii)$$

Multiplying (ii) by k and adding to (i), we get

$$a\frac{x_1+kx_2}{1+k}+b\frac{y_1+ky_2}{1+k}+c\frac{z_1+kz_2}{1+k}+d=0. \qquad \ldots (iii)$$

The relation (iii) shows that the point

$$\left(\frac{x_1+kx_2}{1+k}, \frac{y_1+ky_2}{1+k}, \frac{z_1+kz_2}{1+k}\right)$$

is *also* on the locus. But, for different values of k, these are the general co-ordinates of any point on the line PQ. Thus every point on the straight line joining any two *arbitrary points* on the locus also lies on the locus.

The given equation, therefore, represents a plane.

Hence every equation of the first degree in x, y, z represents a plane.

Ex. Find the co-ordinates of the points where the plane

$$ax+by+cz+d=0$$

meets the three co-ordinate axes.

2·2. Normal form of the equation of a plane. *To find the equation of a plane in terms of p, the length of the normal from the origin to it and l, m, n the direction cosines of that normal ; (p is to be always regarded positive).*

Let OK be the normal from O to the given plane ; K being the foot of the normal.

Then $OK=p$ and l, m, n are its direction cosines.

Take any point $P(x, y, z)$ on the plane.

Now, $PK \perp OK$, for it lies in the plane which is $\perp OK$.

Therefore the projection of OP on $OK = OK = p$.

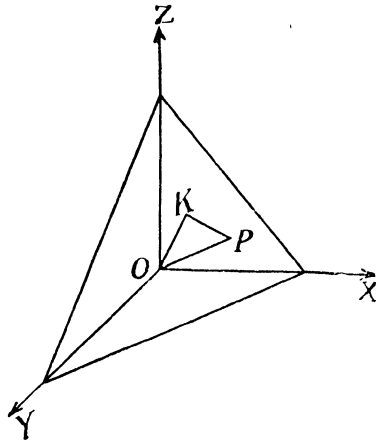

Fig. 12

Also the projection of the line OP joining
$O(0, 0, 0)$ and $P(x, y, z)$,
on the line OK whose direction cosines are
l, m, n,
is

$$l(x-0)+m(y-0)+n(z-0)=lx+my+nz. \quad (\S 1{\cdot}84, \text{ p. } 14)$$

Hence $$\mathbf{lx+my+nz=p.}$$

This equation, being satisfied by the co-ordinates of any point $P(x, y, z)$ on the given plane, represents the plane and is known as the *normal form* of the equation of a plane.

Cor. *The equation of any plane is of the first degree in x, y, z.*

This is the converse of the theorem proved in § 2·1.

Ex. Find the equation of the plane containing the lines through the origin with direction cosines proportional to (1, −2, 2) and (2, 3, −1).

[*Ans.* $4x-5y-7z=0$.

2·3. Transformation to the Normal form. *To transform the equation*

$$ax+by+cz+d=0$$

to the normal form

$$lx+my+nz=p.$$

As these two equations represent the same plane, we have

$$-\frac{d}{p}=\frac{a}{l}=\frac{b}{m}=\frac{c}{n}=\pm\frac{\sqrt{(a^2+b^2+c^2)}}{\sqrt{(l^2+m^2+n^2)}}=\pm\sqrt{(a^2+b^2+c^2)}.$$

Thus, $-d/p=\pm\sqrt{(a^2+b^2+c^2)}$ and as p, according to our convention, is to be always positive, we shall take positive or negative sign with the radical according as, d, is negative or positive.

Thus, if d be positive,

$$l=-\frac{a}{\sqrt{\Sigma a^2}}; \; m=-\frac{b}{\sqrt{\Sigma a^2}}; \; n=-\frac{c}{\sqrt{\Sigma a^2}}; \; p=+\frac{d}{\sqrt{\Sigma a^2}}.$$

If d be negative, we have only to change the signs of all these. Thus the normal form of the equation $ax+by+cz+d=0$ is

$$-\frac{a}{\sqrt{\Sigma a^2}}x-\frac{b}{\sqrt{\Sigma a^2}}y-\frac{c}{\sqrt{\Sigma a^2}}z=\frac{d}{\sqrt{\Sigma a^2}},$$ if d be positive ;

$$\frac{a}{\sqrt{\Sigma a^2}}x+\frac{b}{\sqrt{\Sigma a^2}}y+\frac{c}{\sqrt{\Sigma a^2}}z=-\frac{d}{\sqrt{\Sigma a^2}},$$ if d be negative.

2·31. Direction cosines of normal to a plane. From above we deduce a very important fact that the direction cosines of normal to any plane are proportional to the co-efficients of x, y, z in its equation or, that the direction ratios of the normal to a plane are the co-efficients of x, y, z in its equation.

Thus,

$$a, b, c$$

are the direction ratios of the normal to the plane

$$ax+by+cz+d=0.$$

Ex. 1. Find the direction cosines of the normals to the planes

(*i*) $2x-3y+6z=7$. (*ii*) $x+2y+2z-1=0$.

[*Ans.* (*i*) 2/7, −3/7, 6/7, (*ii*) 1/3, 2/3, 2/3.

Ex. 2. Show that the normals to the planes

$$x-y+z=1,\ 3x+2y-z+2=0$$

are inclined to each other at an angle 90°.

2·32. Angle between two planes. Angle between two planes is equal to the angle between the normals to them from any point. Thus the angle between the two planes

$$ax+by+cz+d=0, \text{ and } a_1x+b_1y+c_1z+d_1=0$$

is equal to the angle between the lines with direction ratios

$$a, b, c,$$
$$a_1, b_1, c_1,$$

and is, therefore,

$$=\cos^{-1}\left(\frac{aa_1+bb_1+cc_1}{\sqrt{(\Sigma a^2)}\ \sqrt{(\Sigma a_1^2)}}\right).$$

2·33. Parallelism and perpendicularity of two planes. Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular. *Thus the two planes*

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0$$

will be parallel, if

$$\mathbf{a/a_1=b/b_1=c/c_1}\ ;$$

and will be perpendicular, if

$$\mathbf{aa_1+bb_1+cc_1=0.}$$

Exercises

1. Find the angles between the planes

(*i*) $2x-y+2z=3,\ 3x+6y+2z=4$. [*Ans.* $\cos^{-1}(4/21)$.

(*ii*) $2x-y+z=6,\ x+y+2z=7$. [*Ans.* $\pi/3$.

(*iii*) $3x-4y+5z=0,\ 2x-y-2z=5$. [*Ans.* $\pi/2$.

2. Show that the equations

$$ax+by+r=0,\ by+cz+p=0,\ cz+ax+q=0$$

represent planes respectively perpendicular to XY, YZ, ZX planes.

3. Show that $ax+by+cz+d=0$ represents planes, perpendicular respectively to YZ, ZX, XY planes, if a, b, c separately vanish. (Similar to Ex. 2).

4. Show that the plane

$$x+2y-3z+4=0$$

is perpendicular to each of the planes

$$2x+5y+4z+1=0,\ 4x+7y+6z+2=0.$$

2·4. Determination of a plane under given conditions. *The general* equation $ax+by+cz+d=0$ of a plane contains *three* arbitrary constants (ratios of the co-efficients a, b, c, d) and, therefore, a plane can be found to satisfy three conditions each giving rise to only one relation between the constants. The three constants can then be determined from the three resulting relations.

We give below a few sets of conditions which determine a plane:—

(*i*) passing through *three* non-collinear points ;

(*ii*) passing through *two* given points and perpendicular to a given plane ;

(*iii*) passing through a given point and perpendicular to *two* given planes.

2·41. Intercept form of the equation of a plane. *To find the equation of a plane in terms of the intercepts a, b, c which it makes on the axes.*

Let the equation of the plane be

$$Ax+By+Cz+D=0. \qquad \ldots(1)$$

The co-ordinates of the point in which this plane meets the X-axis are given to be $(a, 0, 0)$. Substituting these in equation (1), we obtain

$$aA+D=0,$$

or

$$-\frac{A}{D}=\frac{1}{a}.$$

Similarly

$$-\frac{B}{D}=\frac{1}{b}\ ;\quad -\frac{C}{D}=\frac{1}{c}.$$

The equation (1) can be re-written as

$$-\frac{A}{D}x-\frac{B}{D}y-\frac{C}{D}z=1,$$

so that, after substitution, we obtain

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

as the required equation of the plane.

Note. The fact that a plane makes intercepts a, b, c, on the three axes is equivalent to the statement that it passes through the three points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, so that what we have really done here is to determine the three ratios of the co-efficients in (1) in order that the same may pass through these points.

Ex. 1. Find the intercepts of the plane $2x-3y+4z=12$ on the co-ordinate axes. [*Ans.* 6, −4, 3.

Ex. 2. A plane meets the co-ordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c) ; show that the equation of the plane is $x/a+y/b+z/c=3$.

Ex. 3. Prove that a variable plane which moves so that the sum of the reciprocals of its intercepts on the three co-ordinate axes is constant, passes through a fixed point.

2·42. Plane through three points. *To find the equation of the plane passing through the three non-collinear points*

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).$$

Let the required equation of the plane be

$$ax+by+cz+d=0. \qquad \ldots(i)$$

As the given points lie on the plane, we have

$$ax_1+by_1+cz_1+d=0, \qquad \ldots(ii)$$

$$ax_2+by_2+cz_2+d=0, \qquad \ldots(iii)$$

$$ax_3+by_3+cz_3+d=0. \qquad \ldots(iv)$$

Eliminating a, b, c, d from (i)—(iv), we have

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0$$

which is the required equation of the plane.

Note. In actual numerical exercises, the student would find it more convenient to follow the method of the first exercise below.

Exercises

1. *Find the equation of the plane through*

$$P(2, 2, -1), Q(3, 4, 2), R(7, 0, 6).$$

The general equation of a plane through $P(2, 2, -1)$ is

$$a(x-2)+b(y-2)+c(z+1)=0 \quad \text{(Refer 4, § 2·5, } p. 25) \qquad \ldots(i)$$

It will pass through Q and R, if

$$a+2b+3c=0$$

$$5a-2b+7c=0.$$

These give

$$\frac{a}{20}=\frac{b}{8}=\frac{c}{-12} \quad \text{or} \quad \frac{a}{5}=\frac{b}{2}=\frac{c}{-3}$$

Substituting these values in (i), we have

$$5(x-2)+2(y-2)-3(z+1)=0$$

i.e.,

$$5x+2y-3z-17=0$$

as the required equation.

2. Find the equation of the plane through the three points (1, 1, 1), (1, −1, 1), (−7, −3, −5) and show that it is perpendicular to the XZ plane. [*Ans.* $3x-4z+1=0$.

3. Obtain the equation of the plane passing through the point $(-2, -2, 2)$ and containing the line joining the points $(1, 1, 1)$ and $(1, -1, 2)$.

[*Ans.* $x-3y-6z+8=0$.

4. If, from the point $P(a, b, c)$, perpendiculars PL, PM be drawn to YZ and ZX planes, find the equation of the plane OLM. [*Ans.* $bcx+cay-abz=0$.

5. Show that the four points $(-6, 3, 2)$, $(3, -2, 4)$, $(5, 7, 3)$ and $(-13, 17, -1)$ are coplanar.

6. Show that the join of points $(6, -4, 4)$, $(0, 0, -4)$ intersects the join of $(-1, -2, -3)$, $(1, 2, -5)$.

7. Show that $(-1, 4, -3)$ is the circumcentre of the triangle formed by the points $(3, 2, -5)$, $(-3, 8, -5)$, $(-3, 2, 1)$.

8. *Find the equation of the plane through the points* $(2, 2, 1)$ *and* $(9, 3, 6)$ *and perpendicular to the plane*

$$2x+6y+6z=9.$$

Any plane through $(2, 2, 1)$ is

$$a(x-2)+b(y-2)+c(z-1)=0. \quad \ldots (i)$$

It will pass through $(9, 3, 6)$ if

$$a(9-2)+b(3-2)+c(6-1)=0$$

i.e.,

$$7a+b+5c=0 \quad \ldots (ii)$$

The plane (i) will be perpendicular to the given plane if

$$2a+6b+6c=0. \quad \ldots (iii)$$

From (ii) and (iii), we have

$$\frac{a}{-24}=\frac{b}{-32}=\frac{c}{40} \quad \text{or} \quad \frac{a}{3}=\frac{b}{4}=\frac{c}{-5}.$$

Substituting in (i), we see that the equation of the required plane is

$$3(x-2)+4(y-2)-5(z-1)=0$$

or

$$3x+4y-5z=9.$$

9. Show that the equations of the three planes passing through the points, $(1, -2, 4)$, $(3, -4, 5)$ and perpendicular to XY, YZ, ZX planes are $x+y+1=0$; $x-2z+7=0$; $y+2z=6$ respectively.

10. Obtain the equation of the plane through the point $(-1, 3, 2)$ and perpendicular to the two planes $x+2y+2z=5$; $3x+3y+2z=8$.

[*Ans.* $2x-4y+3z+8=0$.

11. Find the equation of the plane through $A(-1, 1, 1)$ and $B(1, -1, 1)$ and perpendicular to the plane $x+2y+2z=5$. [*Ans.* $2x+2y-3z+3=0$.

12. Find the equations of the two planes through the points $(0, 4, -3)$, $(6, -4, 3)$ other than the plane through the origin, which cut off from the axes intercepts whose sum is zero. (*M.T.*)

[*Ans.* $2x-3y-6z=6$; $6x+3y-2z=18$.

13. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

2·5. Systems of planes. The equation of a plane satisfying two conditions will involve one arbitrary constant which can be chosen in an infinite number of ways, thus giving rise to an infinite number of planes, called a *system of planes.*

The arbitrary constant which is different for different members of the system is called a *parameter.*

Similarly the equation of a plane satisfying one condition will involve two parameters.

The following are the equations of a few systems of planes involving one or two arbitrary constants.

1. The equation

$$ax+by+cz+k=0$$

represents a system of planes parallel to the plane

$$ax+by+cz+d=0,$$

k being the parameter. (§ 2·33, *p.* 21).

2. The equation

$$ax+by+cz+k=0$$

represents a system of planes perpendicular to the line with direction ratios a, b, c ; k being the parameter. (§2·31, *p.* 21).

3. The equation

$$(ax+by+cz+d)+k(a_1x+b_1y+c_1z+d_1)=0 \quad \ldots(1)$$

represents a system of planes passing through the line of intersection of the planes

$$ax+by+cz+d=0, \quad \ldots(2)$$

$$a_1x+b_1y+c_1z+d_1=0\ ; \quad \ldots(3)$$

k being the parameter, for

(*i*) the equation, being of the first degree in x, y, z, represents a plane ;

(*ii*) it is evidently satisfied by the co-ordinates of the points which satisfy (2) and (3), whatever value k may have.

4. The system of planes passing through the point (x_1, y_1, z_1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

where the required *two* parameters are the two ratios of the co-efficients A, B, C ; for, the equation is of the first degree and is clearly satisfied by the point (x_1, y_1, z_1), whatever be the values of the ratios of the co-efficients.

Exercises

1. *Find the equation of the plane passing through the intersection of the planes*

$$x+y+z=6 \text{ and } 2x+3y+4z+5=0$$

and the point (1, 1, 1).

The plane

$$x+y+z-6+k(2x+3y+4z+5)=0, \quad \ldots(i)$$

passes through the intersection of the given planes for all values of k.

It will pass through (1, 1, 1) if

$$-3+14k=0 \text{ or } k=3/14.$$

Putting $k=3/14$ in (*i*), we obtain

$$20x+23y+26z-69=0,$$

which is the required equation of the plane.

2. Obtain the equation of the plane through the intersection of the planes

$$x+2y+3z+4=0 \text{ and } 4x+3y+2z+1=0$$

and the origin. [*Ans.* $3x+2y+z=0$.

3. *Find the equation of the plane passing through the line of intersection of the planes*

$$2x-y=0 \text{ and } 3z-y=0$$

and perpendicular to the plane

$$4x+5y-3z=8.$$

The plane

$$2x-y+k(3z-y)=0, \text{ i.e., } 2x-(1-k)y+3kz=0,$$

passes through the line of intersection of the given planes whatever k may be. It will be perpendicular to

$$4x+5y-3z=8,$$

if $$2\cdot4-(1+k)\cdot5+3k(-3)=0, \text{ i.e. } 14k=3.$$

$$\therefore \quad k=\frac{3}{14}.$$

Thus the required equation is

$$2x-y+\left(\frac{3}{14}\right)(3z-y)=0,$$

i.e., $$28x-17y+9z=0.$$

4. Find the equation of the plane which is perpendicular to the plane

$$5x+3y+6z+8=0$$

and which contains the line of intersection of the planes

$$x+2y+3z-4=0,\ 2x+y-z+5=0. \qquad (L.U.\ 1934)$$

[*Ans.* $51x+15y-50z+173=0$.

5. The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$, find the equation of the plane in its new position. [*Ans.* $22x+5y-4z-35=0$.

6. Find the equation of the plane through the intersection of the planes

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0$$

and perpendicular to the XY plane.

[*Ans.* $x(ac_1-a_1c)+y(bc_1-b_1c)+(dc_1-d_1c)=0$.

7. *Obtain the equation of the plane through the point* (x_1, y_1, z_1) *and parallel to the plane* $ax+by+cz+d=0$.

The plane

$$ax+by+cz+k=0$$

is parallel to the given plane for all values of k.

It will pass through (x_1, y_1, z_1), if

$$ax_1+by_1+cz_1+k=0.$$

Subtracting, we get

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0.$$

which is the required equation.

8. Find the equation of the plane through the point (2, 3, 4) and parallel to the plane $5x-6y+7z=3$. [*Ans.* $5x-6y+7z=20$.

9. Find the equation of the plane that passes through (3, −3, 1) and is normal to the line joining the points (3, 4, −1) and (2, −1, 5).

[*Ans.* $x+5y-6z+19=0$.

10. Obtain the equation of the plane that bisects the line joining (1, 2, 3), (3, 4, 5), at right angles.

11. $$x+2y-z-3=0,\ 3x-y+2z-1=0,$$
$$2x-2y+3z-2=0,\ x-y+z+1=0$$

are four planes. Show that the line of intersection of the first two planes is coplanar with the line of intersection of the latter two and find the equation of the plane containing the two lines.

The planes

$$x+2y-z-3+k(3x-y+2z-1)=0$$

and $$2x-2y+3z-2+k'(x-y+z+1)=0$$

i.e., $$(1+3k)x+(2-k)y+(-1+2k)z+(-3-k)=0$$

and $$(2+k')x+(-2-k')y+(3+k')z+(-2+k')=0,$$

separately contain the two lines. The two lines will be coplanar if, for some

values of k and k', they become identical. This requires

$$\frac{1+3k}{2+k'}=\frac{2-k}{-2-k'}=\frac{-1+2k}{3+k'}=\frac{-3-k}{-2+k'},$$

or
$$6+4k+3k'+2kk'=0, \quad \ldots(i)$$
$$4+k+kk'=0, \quad \ldots(ii)$$
$$11-k+2k'+3kk'=0. \quad \ldots(iii)$$

(i) and (ii) give

$$k=-3/2,\ k'=5,$$

and
$$k=2. \qquad k'=-2.$$

Of these two sets of values, $k=-3/2$ and $k'=5$ satisfy (iii) also. Thus the two planes become identical for $k=-3/2$ and $k'=5$. Hence the two lines are coplanar and the equation of the plane containing them is

$$7x-7y+8z+3=0.$$

12. Show that the line of intersection of the planes

$$7x-4y+7z+16=0,\ 4x+3y-2z+3=0$$

is coplanar with the line of intersection of

$$x-3y+4z+6=0,\ x-y+z+1=0.$$

Obtain the equation of the plane through both.

[*Ans.* $3x-7y+9z+13=0$.

13. A variable plane passes through a fixed point (a, b, c) and meets the co-ordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the co-ordinate planes is

$$a/x+b/y+c/z=1.$$

2·6 Two sides of a plane. *Two points*

$$A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$$

lie on the same or different sides of the plane

$$ax+by+cz+d=0,$$

according as the expressions

$$ax_1+by_1+cz_1+d, \qquad ax_2+by_2+cz_2+d$$

are of the same or different signs.

Let the line AB meet the given plane in a point P and let P divide AB in the ratio $r : 1$ so that r is positive or negative according as P divides AB internally or externally, *i.e.*, according as A and B lie on the opposite or the same side of the plane.

Since the point P whose co-ordinates are

$$\left(\frac{rx_2+x_1}{r+1}, \frac{ry_2+y_1}{r+1}, \frac{rz_2+z_1}{r+1}\right)$$

lies on the given plane, therefore

$$a\frac{rx_2+x_1}{r+1}+b\frac{ry_2+y_1}{r+1}+c\frac{rz_2+z_1}{r+1}+d=0,$$

or
$$r(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0,$$

or
$$r=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

This shows that r is negative or positive according as

$$ax_1+by_1+cz_1+d,\ ax_2+by_2+cz_2+d$$

are of the same or different signs.

Thus the theorem is proved.

Ex. Show that the origin and the point (2, −4, 3) lie on different sides of the plane $x+3y-5z+7=0$.

2·7. Length of the perpendicular from a point to a plane. *To find the perpendicular distance of the point*

$$P(x_1, y_1, z_1)$$

from the plane

$$lx+my+nz=p.$$

The equation of the plane through $P(x_1, y_1, z_1)$ parallel to the given plane is

$$lx+my+nz=p_1,$$

where $$lx_1+my_1+nz_1=p_1.$$

Let OKK' be the perpendicular from the origin O to the two parallel planes meeting them in K and K' so that

$$OK=p \text{ and } OK'=p_1.$$

Draw $PL \perp$ given plane.

Then $$LP=OK'-OK$$
$$=p_1-p=lx_1+my_1+nz_1-p.$$

Cor. *To find the length of the perpendicular from* (x_1, y_1, z_1) *to the plane* $ax+by+cz+d=0$.

The normal form of the given equation of the plane being

$$\pm\frac{a}{\sqrt{\Sigma a^2}}x \pm\frac{b}{\sqrt{\Sigma a^2}}y \pm\frac{c}{\sqrt{\Sigma a^2}}z \pm\frac{d}{\sqrt{\Sigma a^2}}=0,$$

the required length of the perpendicular is

$$\pm\frac{\mathbf{ax_1+by_1+cz_1+d}}{\sqrt{(\mathbf{a^2+b^2+c^2})}}.$$

Thus the length of the perpendicular from (x_1, y_1, z_1) *to the plane*

$$ax+by+cz+d=0$$

is obtained by substituting

$$x_1, y_1, z_1, \text{ for } x, y, z,$$

respectively in the expression,

$$ax+by+cz+d,$$

and dividing the same by

$$\sqrt{(a^2+b^2+c^2)}.$$

Exercises

1. Find the distances of the points (2, 3, 4) and (1, 1, 4) from the plane

$$3x-6y+2z+11=0. \qquad [Ans.\ 1\ ;\ 16/7.$$

2. Show that the distance between the parallel planes

$$2x-2y+z+3=0 \text{ and } 4x-4y+2z+5=0$$

is 1/6.

(The distance between two parallel planes is the distance of any point on one from the other).

3. Find the locus of the point whose distance from the origin is three times its distance from the plane $2x-y+2z=3$.

$$[Ans.\ 3x^2+3z^2-4xy+8xz-4yz-12x+6y-12z+9=0.$$

4. Show that (1/8, 1/8, 1/8) is in the centre of the tetrahedron formed by the four planes $x=0$, $y=0$, $z=0$, $x+2y+2z=1$.

5. Sum of the distances of any number of fixed points from a variable plane is zero ; show that the plane passes through a fixed point.

6. A variable plane which remains at a constant distance, $3p$, from the origin cuts the co-ordinate axes at A, B, C. Show that the locus of centroid of the triangle ABC is

$$x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

2·71. Bisectors of angles between two planes. *To find the equations of the bisectors of the angles between the planes*

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0.$$

If (x, y, z) be any point on any one of the planes bisecting the angles between the planes, then the perpendiculars from this point to the two planes must be equal (in magnitude).

Hence

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\pm\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

are the equations of the two bisecting planes.

Of these two bisecting planes, one bisects the acute and the other the obtuse angle between the given planes.

The bisector of the acute angle makes with either of the planes an angle which is less than 45° and the bisector of the obtuse angle makes with either of them an angle which is greater than 45°. This gives a test for determining which angle, acute or obtuse, each bisecting plane bisects.

Ex. *Find the equations of the planes bisecting the angles between the planes*

$$x+2y+2z-3=0, \qquad \ldots(i)$$

$$3x+4y+12z+1=0. \qquad \ldots(ii)$$

and specify the one which bisects the acute angle.

The equations of the two bisecting planes are

$$\frac{x+2y+2z-3}{3}=\pm\frac{3x+4y+12z+1}{13}$$

or
$$2x+7y-5z-21=0, \qquad \ldots(iii)$$

and
$$11x+19y+31z-18=0. \qquad \ldots(iv)$$

If θ be the angle between the planes (i) and (iii), we have

$$\cos\theta=2/\sqrt{78}$$

so that $\tan\theta=\sqrt{74/2}$, which being greater than 1, we see that θ is greater than 45°. Hence (iii) bisects the obtuse angle, and consequently, (iv) bisects the acute angle.

Note. Sometimes we distinguish between the two bisecting planes by finding that plane which bisects the angle between the given planes containing the origin. To do this, we express the equations of the given planes so that d and d_1 are positive. Consider the equation

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}} \qquad \ldots(A)$$

Since, by virtue of the equality (A), the expressions $ax+by+cz+d$ and $a_1x+b_1y+c_1z+d_1$ must have the same sign (denominators being both positive), the points (x, y, z) on the locus lie on the origin or the non-origin side of both the planes, *i.e.*, the points on the locus lie in the angle between the planes containing the origin. Thus the equation (A) represents the plane bisecting that angle between the planes which contains the origin.

Similarly,

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}}=-\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1{}^2+b_1{}^2+c_1{}^2)}}$$

represents the plane bisecting the other angle between the given planes.

Exercises

1. Find the bisector of the acute angle between the planes

$$2x-y+2z+3=0,\quad 3x-2y+6z+8=0.$$

[*Ans.* $23x-13y+32z+45=0$.

2. Show that the plane

$$14x-8y+13=0$$

bisects the obtuse angle between the planes

$$3x+4y-5z+1=0,\ 5x+12y-13z=0.$$

3. Find the bisector of that angle between the planes

$$3x-6y+2z+5=0,\ 4x-12y+3z-3=0.$$

which contains the origin. [*Ans.* $67x-162y+47z+44=0$.

2·8. Joint equation of two planes. *To find the condition so that the homogeneous second degree equation*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(1)$$

may represent two planes.

Let the two planes represented by (1) be

$$lx+my+nz=0, \text{ and } l'x+m'y+n'z=0.$$

There cannot appear constant terms in the equations of the planes, for, otherwise, their joint equation will not be homogeneous.

We have

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy\equiv(lx+my+nz)(l'x+m'y+n'z)$$

so that comparing co-efficients, we obtain

$$a=ll',\ b=mm',\ c=nn'$$

and $$2f=m'n+mn',\ 2g=ln'+l'n,\ 2h=lm'+l'm.$$

In order to find the required condition, we have to eliminate $l,\ m,\ n\ ;\ l'\ m',\ n'$ from the above six relations and this can be easily effected as follows. We have

$$0=\begin{vmatrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{vmatrix}\times\begin{vmatrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{vmatrix}$$

$$=\begin{vmatrix} ll' & +l'l, & l'm+lm', & l'n+ln' \\ lm' & +l'm, & mm'+m'm, & m'n+mn' \\ n'l & +nl', & n'm+nm', & n'n+nn' \end{vmatrix}$$

$$=8\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=8(abc+2fgh-af^2-bg^2-ch^2)$$

Hence

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0}$$

is the required condition.

Cor. Angle between planes. If θ be the angle between the planes represented by (1), we have

$$\tan\theta=\frac{\sqrt{[(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2]}}{ll'+mm'+nn'}$$

$$=\frac{2\sqrt{(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}.$$

The planes will be at right angles if $\mathbf{a+b+c=0}$, for then θ is 90°.

Ex. Show that the following equations represent pairs of planes and also find the angles between each pair.

(*i*) $12x^2-2y^2-6z^2-2xy+7yz+6zx=0$. [*Ans.* $\cos^{-1}(4/21)$.

(*ii*) $2x^2-2y^2+4z^2+6xz+2yz+3xy=0$. [*Ans.* $\cos^{-1}(4/9)$.

2·9. Orthogonal projection on a plane. Determination of Plane Areas. Def. *The foot of the perpendicular drawn from any point P to a given plane, π, is called the orthogonal projection of the point P on the plane π.*

This plane, π, is called the plane of the projection.

Thus (Fig. 1, *p.* 1) L, M, N are respectively the orthogonal projections of the point P on the YZ, ZX and XY planes.

The projection of a curve on the plane of projection is the locus of the projection on the plane of any point on the curve.

The projection of the area enclosed by a plane curve is the area enclosed by the projection of the curve on the plane of projection.

In particular, the projection of a straight line is the locus of the foot of the perpendicular drawn from any point on it to the plane of the projection.

2·91. The following simple results of *Pure solid geometry* are assumed without proof:—

(1) The projection of a straight line is a straight line.

(2) If a line AB in a plane, be perpendicular to the line of intersection of this plane with the plane of projection, then the length of its projection is $AB\cos\theta$; θ being the angle between the two planes.

In case AB is parallel to the plane of projection, then the length of the projection is the same as that of AB.

(3) The projection of the area, A, enclosed by any curve in a plane is $A\cos\theta$; θ being the angle between the plane of the area and the plane of projection.

Theorem. *If A_x, A_y, A_z be the areas of the projections of an area, A, on the three co-ordinate planes, then*

$$A^2=A_x^2+A_y^2+A_z^2$$

Let l, m, n be the direction cosines of the normal to the plane of the area A.

Since l is the cosine of the angle between the YZ plane and the plane of the area A, therefore

$$A_x = l.A.$$

Similarly, $$A_y = m.A,$$

and $$A_z = n.A.$$

Hence $$A_x^2 + A_y^2 + A_z^2 = A^2(l^2 + m^2 + n^2) = A^2.$$

Exercises

1. *Find the area of the triangle whose vertices are the points*

$(1, 2, 3), (-2, 1, -4), (3, 4, -2).$ *(D.U. Hons., 1947)*

To find the area A of this triangle, we find the areas A_x, A_y, A_z, of the projection of the same on the co-ordinate planes.

The vertices of the projection of the triangle on the XY plane are

$$(1, 2, 0), (-2, 1, 0), (3, 4, 0),$$

so that

$$A_x = \frac{1}{2}\begin{vmatrix} 1, & 2, & 1 \\ -2, & 1, & 1 \\ 3, & 4, & 1 \end{vmatrix} = -2.$$

Similarly, $$A_y = \frac{1}{2}\begin{vmatrix} 1, & 3, & 1 \\ -2, & -4, & 1 \\ 3, & -2, & 1 \end{vmatrix} = \frac{29}{2}$$

and $$A_z = \frac{1}{2}\begin{vmatrix} 2, & 3, & 1 \\ 1, & -4, & 1 \\ 4, & -2, & 1 \end{vmatrix} = \frac{19}{2}.$$

Therefore, the area of the triangle

$$= \sqrt{(A_x^2 + A_y^2 + A_z^2)} = \sqrt{\left(4 + \frac{(29)^2}{4} + \frac{(19)^2}{4}\right)} = \frac{\sqrt{1218}}{2}.$$

2. Find the areas of the triangles whose vertices are the points

(*i*) $(a, 0, 0), (0, b, 0), (0, 0, c)$.

(*ii*) $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$.

3. From a point $P(x', y', z')$, a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B, C ; prove that the area of the triangle ABC is $r^5/2x'y'z'$, where r is the measure of OP.

2·10. Volume of a tetrahedron. *To find the volume of a tetrahedron in terms of the co-ordinates*

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$$

of its vertices A, B, C, D.

Let V be the volume of the tetrahedron $ABCD$.

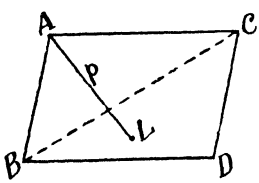

Fig. 13

Then

$$V = \tfrac{1}{3}p\triangle, \qquad \ldots(i)$$

where p is the length of the perpendicular AL from any vertex A to the opposite face BCD ; and $\triangle$ is the area of the triangle BCD.

The equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0$$

or
$$x\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix} - y\begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix} + z\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix} = 0.$$

$\therefore$ the length of the perpendicular, $p =$

$$\frac{x_1\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix} - \begin{vmatrix} x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ x_4, & y_4, & z_4 \end{vmatrix}}{\left\{\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & z_4, & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}} \quad \ldots(ii)$$

The numerator of $p = \begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$.

If $\triangle_x$, $\triangle_y$ $\triangle_z$ be the areas of the projections of $\triangle$ on the YZ ZX, XY planes respectively, we obtain

$$2\triangle_x = \begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}, \quad 2\triangle_y = \begin{vmatrix} x_2, & z_2, & 1 \\ x_3, & z_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix},$$

$$2\triangle_z = \begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix},$$

Therefore, denominator of $p=[4(\triangle_x{}^2+\triangle_y{}^2+\triangle_z{}^2)]^{\frac{1}{2}}=2\triangle$.

From (i) and (ii), we see that the required volume

$$=\tfrac{1}{3}\triangle\ p=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

Exercises

1. The vertices of a tetrahedron are (0, 1, 2), (3, 0, 1), (4, 3, 6), (2, 3, 2); show that its volume is 6.

2. A, B, C are three fixed points and a variable point P moves so that the volume of the tetrahedron $PABC$ is constant; show that the locus of the point P is a plane parallel to the plane ABC.

3. A variable plane makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Find

(i) the locus of the centroid of the tetrahedron. [*Ans.* $xyz=6k^3$.

(ii) the locus of the foot of the perpendicular from the origin to the plane. [*Ans.* $(x^2+y^2+z^2)^3=384k^3xyz$.

4. *Find the volume of the tetrahedron in terms of three edges which meet in a point and of the angles which they make with each other.* (*P.U. 1939*)

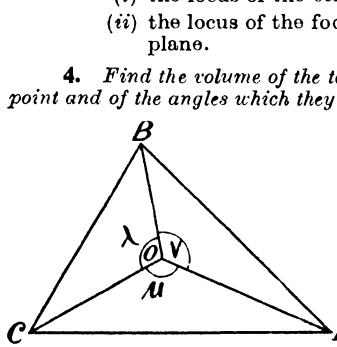

Fig. 14

Let $OABC$ be a tetrahedron.

Let

$$OA=a,\ OB=b,\ OC=c.$$

Let

$$\angle BOC=\lambda,\ \angle COA=\mu.$$

and $$\angle AOB=\nu.$$

We take O as origin and any system of three mutually perpendicular lines through O as co-ordinate axes. Let the direction cosines of OA, OB, OC be

$$l_1, m_1, n_1\ ;\ l_2, m_2, n_2\ ;\ l_3, m_3, n_3.$$

Therefore, the co-ordinates of A, B, C are

$$(l_1a, m_1a, n_1a)\ ;\ (l_2b, m_2b, n_2b)\ ;\ (l_3c, m_3c, n_3c) \qquad (\S\ 1{\cdot}61)$$

Therefore, the volume of the tetrahedron $OABC$

$$=\frac{1}{6}\begin{vmatrix} 0, & 0, & 0, & 1 \\ l_1a, & m_1a, & n_1a, & 1 \\ l_2b, & m_2b, & n_2b, & 1 \\ l_3c, & m_3c, & n_3c, & 1 \end{vmatrix}=\frac{1}{6}\begin{vmatrix} l_1a, & m_1a, & n_1a \\ l_2b, & m_2b, & n_2b \\ l_3c, & m_3c, & n_3c \end{vmatrix}=\frac{abc}{6}\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

Now

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2=\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}\times\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1^2, & \Sigma l_1 l_2, & \Sigma l_1 l_3 \\ \Sigma l_1 l_2, & \Sigma l_2^2, & \Sigma l_2 l_3 \\ \Sigma l_3 l_1, & \Sigma l_3 l_2, & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}$$

Therefore, the volume of the tetrahedron $OABC$

$$= \frac{abc}{6} \begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}$$

5. *Show that the volume of the tetrahedron, the equations of whose faces are*

$$a_r x + b_r y + c_r z + d_r = 0, \; r = (1, 2, 3, 4)$$

is

$$\frac{\Delta^3}{6D_1 D_2 D_3 D_4}$$

where Δ *is the determinant*

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}$$

and D_1, D_2, D_3, D_4 *are the co-factors of* d_1, d_2, d_3, d_4 *respectively in the determinant* Δ.

Let (x_1, y_1, z_1) be the point of intersection of the three planes

$$a_r x + b_r y + c_r z + d_r = 0, \; r = (2, 3, 4),$$

so that (x_1, y_1, z_1) is one of the vertices of the tetrahedron.

Let (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the other vertices, similarly obtained.

We write

$$a_1 x_1 + b_1 y_1 + c_1 z_1 + d_1 = k_1$$

i.e.,

$$a_1 x_1 + b_1 y_1 + c_1 z_1 + (d_1 - k_1) = 0. \quad \ldots (1)$$

Also, we have

$$a_2 x_1 + b_2 y_1 + c_2 z_1 + d_2 = 0, \quad \ldots (2)$$
$$a_3 x_1 + b_3 y_1 + c_3 z_1 + d_3 = 0, \quad \ldots (3)$$
$$a_4 x_1 + b_4 y_1 + c_4 z_1 + d_4 = 0. \quad \ldots (4)$$

Eliminating x_1, y_1, z_1 from (1), (2), (3), (4), we have

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 - k_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = 0,$$

or

$$\Delta + k_1 \begin{vmatrix} a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \\ a_4, & b_4, & c_4 \end{vmatrix} = 0,$$

or $$\Delta - k_1 D_1 = 0.$$

$\therefore$ $$k_1 = \frac{\Delta}{D_1}.$$

Similarly

$$a_2x_2+b_2y_2+c_2z_2+d_2=k_2=\frac{\Delta}{D_2}.$$

$$a_3x_3+b_3y_3+c_3z_3+d_3=k_3=\frac{\Delta}{D_3}$$

$$a_4x_4+b_4y_4+c_4z_4+d_4=k_4=\frac{\Delta}{D_4}$$

We, now, have

$$\begin{vmatrix} a_1, b_1, c_1, d_1 \\ a_2, b_2, c_2, d_2 \\ a_3, b_3, c_3, d_3 \\ a_4, b_4, c_4, d_4 \end{vmatrix} \times \begin{vmatrix} x_1, y_1, z_1, 1 \\ x_2, y_2, z_2, 1 \\ x_3, y_3, z_3, 1 \\ x_4, y_4, z_4, 1 \end{vmatrix} = \begin{vmatrix} k_1, 0, 0, 0 \\ 0, k_2, 0, 0 \\ 0, 0, k_3, 0 \\ 0, 0, 0, k_4 \end{vmatrix}$$

$$= k_1k_2k_3k_4$$

$$= \frac{\Delta^4}{D_1D_2D_3D_4},$$

Therefore the required volume $= \frac{\Delta^3}{6D_1D_2D_3D_4}$

6. Find the volume of the tetrahedron formed by planes whose equations are

$$y+z=0,\ z+x=0,\ x+y=0 \text{ and } x+y+z=1. \qquad (P.\ U.\ 1942)$$

[*Ans.* 2/3.

CHAPTER III

RIGHT LINE

3·1. Equations of a line. A line may be determined as the intersection of any two planes through it.

Now, if

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0$$

be the equations of any two planes through the given line, then these two equations, taken together, give the equations of the line. This follows from the fact that any point on the line lies on both these planes and, therefore, its co-ordinates satisfy both the equations and *conversely*, any point whose co-ordinates satisfy the two equations lies on both these planes, and, therefore, on the line.

Thus, *a straight line in space is represented by* **two** *equations of the first degree in* x, y, z.

Of course any given line can be represented by *different pairs* of first degree equations, for we may take *any* pair of planes through the line and the equations of the same will constitute the equations of the line.

In particular, as the X-axis is the intersection of the XZ and XY planes, its equations are $y=0, z=0$ taken together. Similarly the equations of the Y-axis are $x=0, z=0$ and of the Z-axis are $x=0, y=0$.

Ex. Find the intersection of the line

$$x-2y+4z+4=0,\ x+y+z-8=0$$

with the plane

$$x-y+2z+1=0. \qquad [Ans.\ (2, 5, 1)$$

3·11. Symmetrical form of the equations of a line. *To find the equations of the line passing through a given point* $A(x_1, y_1, z_1)$, *and having direction cosines,* l, m, n.

Let $P\ (x, y, z)$ be any point on the line and let $AP=r$.

Projecting AP on the co-ordinate axes, we obtain

$$x-x_1=lr,\ y-y_1=mr,\ z-z_1=nr \qquad \ldots(i)$$

so that for all points (x, y, z) on the given line,

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r.$$

Thus

$$\frac{\mathbf{x-x_1}}{\mathbf{l}}=\frac{\mathbf{y-y_1}}{\mathbf{m}}=\frac{\mathbf{z-z_1}}{\mathbf{n}} \qquad \ldots(ii)$$

are the *two* required equations of the line.

Clearly, the equations (ii) of the line are not altered if we replace the direction cosines l, m, n by three numbers proportional to them.

so that it suffices to use direction ratios in place of direction cosines while writing down the equations of a line.

Cor. From the relations (*i*), we have

$$\mathbf{x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr,}$$

which are the general co-ordinates of any point on the line in terms of the *parameter r*.

Any value of r will give *some* point on the line and *any* point on the line arises from *some* value of r.

It should be noted, that it is only when l, m, n are the actual direction cosines that r gives the distance between the points (x_1, y_1, z_1) and (x, y, z).

Note 1. The symmetrical form (*ii*) of the equations of a straight line proves useful when we are concerned with the direction cosines of the line or when we wish to obtain the general co-ordinates of any point on the line in terms of a parameter.

Note 2. The equation

$$\frac{x-x_1}{l} = \frac{y-y_1}{m}$$

of first degree, being free of z, represents a plane through the line drawn perpendicular of the XOY plane. Similar statements may be made about the equations

$$\frac{y-y_1}{m} = \frac{z-z_1}{n}, \quad \frac{z-z_1}{n} = \frac{x-x_1}{l}.$$

The equations

$$(x-x_1)/l = (y-y_1)/m,\ (y-y_1)/m = (z-z_1)/n$$

represent a pair of planes through the given line.

3·12. Line through two points. *To find the equations of the line through the two points*

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2).$$

Since

$$x_2 - x_1,\ y_2 - y_1, z_2 - z_1$$

are proportional to the direction cosines of the line, the required equations are

$$\frac{\mathbf{x-x_1}}{\mathbf{x_2-x_1}} = \frac{\mathbf{y-y_1}}{\mathbf{y_2-y_1}} = \frac{\mathbf{z-z_1}}{\mathbf{z_2-z_1}}.$$

Note. Results obtained in Cor. 2 page 6 may be regarded as the parametric equations of the line through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2); λ being the parameter.

Exercises

1. Find k so that the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$

$$\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$$

may be perpendicular to each other. [*Ans.* $-10/7$.

2. Find two points on the line

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z+5}{2}$$

on either side of $(2, -3, -5)$ and at a distance 3 from it.

[*Ans.* $(3, -5, -3)$; $(1, -1, -7)$.

3. *Find the co-ordinates of the point of intersection of the line*

$$\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}$$

with the plane

$$3x+4y+5z=5.$$

Let

$$\frac{x+1}{1}=\frac{y+3}{3}=\frac{z-2}{-2}=r,$$

so that the point

$$r-1,\ 3r-3,\ -2r+2$$

lies on the given line for all values of r.

If it also lies on the given plane, we have

$$3r-3+12r-12-10r+10=5 \quad \text{or} \quad r=2.$$

Hence the required point of intersection is $(1, 3, -2)$.

Its distance from the point $(-1, -3, 2)$ is $\sqrt{56}$ which is different from the value 2 of r. (Why ?)

4. Find the point where the line joining $(2, -3, 1)$, $(3, -4, -5)$ cuts the plane $2x+y+z=7$. [*Ans.* $(1, -2, 7)$.

5. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{1}{3}(x-2)=\frac{1}{4}(y+1)=\frac{1}{12}(z-2)$ and the plane

$$x-y+z=5. \qquad (P.U.\ 1934)\ [Ans.\ 13.$$

6. Find the distance of the point $(3, -4, 5)$ from the plane

$$2x+5y-6z=16$$

measured along a line with direction cosines proportional to $(2, 1, -2)$.

[*Ans.* 60/7.

7. *Find the image of the point* P $(1, 3, 4)$ in the plane

$$2x-y+z+3=0.$$

If two points P, Q be such that the line is bisected perpendicularly by a plane, then either of the points is the image of the other in the plane.

The line through P perpendicular to the given plane is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1},$$

so that the co-ordinates of Q are of the form

$$(2r+1,\ -r+3,\ r+4)$$

Making use of the fact that the mid point

$$(r+1,\ -\tfrac{1}{2}r+3,\ \tfrac{1}{2}r+4).$$

of PQ lies on the given plane, we see that

$$r=-2$$

so that the image of P is $(-3, 5, 2)$.

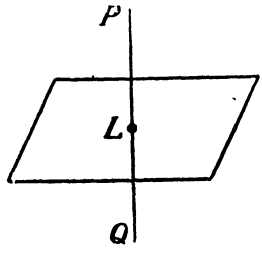

Fig. 15

8. Find the equations to the line through $(-1, 3, 2)$ and perpendicular to the plane $x+2y+2z=3$, the length of the perpendicular and the co-ordinates of its foot. [*Ans.* 2 ; $(-5/3, 5/3, 2/3)$.

9. Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x+3y-4z+1=0$; also find the co-ordinates of the point which is the image of the origin in the plane. (*P.U. Supp.*)

[*Ans.* $(-2/29, -3/29, 4/29)$; $(-4/29, -6/29, 8/29)$.

10. Find the equations to the line through (x_1, y_1, z_1) perpendicular to the plane $ax+by+cz+d=0$ and the co-ordinates of its foot. Deduce the expression for the perpendicular distance of the given point from the given plane.

[*Ans.* $(ar+x_1, br+y_1, cr+z_1)$ where $r=-(ax_1+by_1+cz_1+d)/(a^2+b^2+c^2)$.

11. Show that the line

$$\tfrac{1}{2}(x-7) = -(y+3) = (z-4)$$

intersects the planes

$$6x+4y-5z=4 \text{ and } x-5y+2z=12$$

in the same point and deduce that the line is co-planar with the line of intersection of the planes.

12. Show that the line

$$(x-3)/3 = (2-y)/4 = (z+1)/1$$

intersects the line

$$x+2y+3z=0,\ 2x+4y+3z+3=0.$$

Find their point of intersection. [*Ans.* $(9, -6, 1)$.

13. Show that the equations to the straight line through (a, b, c) parallel to the X-axis are $(x-a)/1 = (y-b)/0 = (z-c)/0$.

14. Show that

$$(x-a)/l = (y-b)/m = (z-c)/0$$

is a straight line perpendicular to the Z-axis.

15. Show that the straight line

$$(x-\alpha)/l = (y-\beta)/m = (z-\gamma)/n$$

meets the locus of the equation

$$ax^2+by^2+cz^2=1,$$

in two points.

Deduce the conditions for the two points to coincide at (α, β, γ).

[*Ans.* $al\alpha+bm\beta+cn\gamma=0$; $a\alpha^2+b\beta^2+c\gamma^2=1$.

16. P is any point on the plane $lx+my+nz=p$ and a point Q is taken on the line OP such that $OP.OQ=p^2$; show that the locus of Q is

$$p(lx+my+nz) = x^2+y^2+z^2.$$

17. A variable plane makes intercepts on the co-ordinate axes the sum of whose squares is constant and equal to k^2. Find the locus of the foot of the perpendicular from the origin to the plane.

[*Ans.* $(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)^2=k^2$.

18. Show that the equations of the lines bisecting the angles between the lines

$$\frac{x-3}{2}=\frac{y+4}{-1}=\frac{z-5}{-2},\quad \frac{x-3}{4}=\frac{y+4}{-12}=\frac{z-5}{3}$$

are

$$\frac{x-3}{38}=\frac{y+4}{-49}=\frac{z-5}{-17},\quad \frac{x-3}{14}=\frac{y+4}{23}=\frac{z-5}{-35}.$$

3·13. It has been seen in §§ 3·11, 3·12, that the equations of a straight line which we generally employ are of two forms.

One is the *symmetrical* form deduced from the consideration that a straight line is completely determined when we know its direction and the co-ordinates of any one point on it, or when any two points on the line are given.

The second form is *unsymmetrical* and is deduced from the consideration that a straight line is the locus of points common to any two planes through it.

In the next section it will be seen how one form of equations can be transformed into the other.

3·14. Transformation from unsymmetrical to the symmetrical form. *To transform the equations*

$$ax+by+cz+d=0,\ a_1x+b_1y+c_1z+d_1=0$$

of a line to the symmetrical form.

To transform these to the symmetrical form, we require

(*i*) the direction ratios of the line, and

(*ii*) the co-ordinates of any one point on it.

Let l, m, n be the direction ratios of the line. Since the line lies in both the planes

$$ax+by+cz+d=0 \text{ and } a_1x+b_1y+c_1z+d_1=0,$$

it is perpendicular to the normals to both of them. As the direction ratios of the normals to the two planes are

$$a, b, c\ ;\ a_1, b_1, c_1,$$

we have

$$al+bm+cn=0,$$
$$a_1l+b_1m+c_1n=0.$$

$$\therefore \quad \frac{l}{bc_1-b_1c}=\frac{m}{ca_1-c_1a}=\frac{n}{ab_1-a_1b}.$$

Now, we require the co-ordinates of *any one* point on the line and there is an infinite number of points from which to choose. We, for the sake of convenience, find the point of intersection of the line with the plane $z=0$. This point which is given by the equations

$$ax+by+d=0 \text{ and } a_1x+b_1y+d_1=0,$$

is

$$\left(\frac{bd_1-b_1d}{ab_1-a_1b}, \frac{a_1d-ad_1}{ab_1-a_1b}, 0\right).$$

Thus, in the symmetrical form, the equations of the given line are

$$\frac{x-(bd_1-b_1d)/(ab_1-a_1b)}{bc_1-b_1c}=\frac{y-(a_1d-ad_1)/(ab_1-a_1b)}{ca_1-c_1a}=\frac{z-0}{ab_1-a_1b}.$$

Exercises

1. Find, in a symmetrical form, the equations of the line

$$x+y+z+1=0,\ 4x+y-2z+2=0$$

and find its direction cosines. (*P.U. 1937*)

[*Ans.* $\frac{x+1/3}{1}=\frac{y+2/3}{-2}=\frac{z}{1}$; $\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

2. Obtain the symmetrical form of the equations of the line

$$x-2y+3z=4,\ 2x-3y+4z=5.$$

[*Ans.* $(x+2)=\frac{1}{2}(y+3)=z$.

3. Find out the points of intersection of the line

$$x+y-z+1=0=14x+9y-7z-1$$

with the XY and YZ planes, and hence put down the symmetrical form of its equations. [*Ans.* $-(x)/2=(y-4)/7=(z-5)/5$.

4. Find the equation of the plane through the point (1, 1, 1) and perpendicular to the line

$$x-2y+z=2,\ 4x+3y-z+1=0.$$

[*Ans.* $x-5y-11z+15=0$.

5. Find the equations of the line through the point (1, 2, 4) parallel to the line

$$3x+2y-z=4,\ x-2y-2z=5.$$

[*Ans.* $(x-1)/6=(2-y)/5=(z-4)/.8$

6. Find the angle between the lines in which the planes

$$3x-7y-5z=1,\ 5x-13y+3z+2=0$$

cut the plane

$$8x-11y+2z=0 \qquad [Ans.\ 90°$$

7. *Find the angle between the lines*

$$3x+2y+z-5=0=x+y-2z-3,$$
$$2x-y-z=0=7x+10y-8z. \qquad (L.U.)\ [Ans.\ 90°.$$

8. Show that the condition for the lines

$$x=az+b,\ y=cz+d\ ;\ x=a_1z+b_1,\ y=c_1z+d_1,$$

to be perpendicular is

$$aa_1+cc_1+1=0.$$

9. What are the symmetrical forms of the equations of the lines

(*i*) $y=b,\ z=c$ $\qquad [Ans.\ x/1=(y-b)/0=(z-c)/0.$

(*ii*) $x=a,\ by+cz=d$ $\qquad [Ans.\ (x-a)/0=(y-d/b)/c=z-b.$

3·2. *To find the angle between*

the line $$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

and the plane $$ax+by+cz+d=0.$$

Angle beween a line and a plane is the complement of the angle between the line and the normal to the plane.

Since the direction cosines of the normal to the given plane and of the given line are proportional to a, b, c and l, m, n respectively, we have

$$\sin\theta=\frac{al+bm+cn}{\sqrt{(a^2+b^2+c^2)}\ \sqrt{(l^2+m^2+n^2)}},$$

where θ is the required angle.

The straight line is parallel to the plane, if $\theta=0$

i.e., $$al+bm+cn=0,$$

which is also evident from the fact *that if a line be parallel to a plane, it is perpendicular to the normal to it.*

Exercises

1. Show that the line $\frac{1}{3}(x-2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4)$ is parallel to the plane $2x+y-2z=3$.

2. Find the equations of the line through the point $(-2, 3, 4)$, and parallel to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$.

$[Ans.\ (x+2)=-\frac{1}{2}(y-3)=(z-4).$

[*Hint.* The direction ratios, l, m, n, of the line are given by the relations $2l+3m+4n=0=3l+4m+5n.$]

3. Find the equation of the plane through the points

$$(1, 0, -1),\ (3, 2, 2)$$

and parallel to the line

$$(x-1)=(1-y)/2=(z-2)/3. \qquad [Ans.\ 4x-y-2z=6.$$

4. Show that the equation of the plane parallel to the join of

$$(3, 2, -5) \text{ and } (0, -4, -11)$$

and passing through the points

$$(-2, 1, -3) \text{ and } (4, 3, 3)$$

is

$$4x+3y-5z=10.$$

5. Find the equation of the plane containing the line

$$2x-5y+2z=6,\ 2x+3y-z=5$$

and parallel to the line $x=-y/6=z/7$. [*Ans.* $6x+y-16=0$.

6. Prove that the equation of the plane through the line

$$u_1\equiv a_1x+b_1y+c_1z+d_1=0,\ u_2\equiv a_2x+b_2y+c_2z+d_2=0$$

and parallel to the line

$$x/l=y/m=z/n$$

is

$$u_1(a_2l+b_2m+c_2n)=u_2(a_1l+b_1m+c_1n). \qquad (D.U.\ Hons.\ 1957)$$

7. Find the equation of the plane through the point $(f, g, h,)$ and parallel to the lines $x/l_r=y/m_r=z/n_r$; $r=1, 2$. [*Ans.* $\Sigma(x-f)(m_1n_2-m_2n_1)=0$.

8. Find the equations of the two planes through the origin which are parallel to the line

$$(x-1)/2=-(y+3)=-(z+1)/2$$

and distant 5/3 from it ; show that the two planes are perpendicular.

[*Ans.* $2x+2y+z=0,\ x-2y+2z=0$.

3·3. Conditions for a line to lie in a plane. *To find the conditions that the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

may lie in the plane

$$ax+by+cz+d=0.$$

The line would lie in the given plane, if, and only if, the co-ordinates

$$lr+x_1,\ mr+y_1,\ nr+z_1$$

of any point on the line satisfy the equation of the plane for all values of r so that

$$r(al+bm+cn)+(ax_1+by_1+cz_1+d)=0,$$

is an identity.

This gives

$$\mathbf{al+bm+cn=0},$$

$$\mathbf{ax_1+by_1+cz_1+d=0}\ ;$$

which are the required *two* conditions.

These conditions lead to the geometrical facts that a line will lie in a given plane, if

(*i*) *the normal to the plane is perpendicular to the line,*

and (*ii*) *any one point on the line lies in the plane.*

Cor. *General equation of the plane containing the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

where

$$Al+Bm+Cn=0. \qquad \ldots (1)$$

Here, $A : B : C$ are the parameters subjected to the condition (1).

Exercises

1. Show that the line $x+10=(8-y)/2=z$ lies in the plane

$$x+2y+3z=6$$

and the line $\frac{1}{3}(x-2)=-(y+2)=\frac{1}{4}(z-3)$ in the plane

$$2x+2y-z+3=0.$$

2. *Find the equation to the plane through the point* (x_1, y_1, z_1) *and through the line*

$$(x-a)/l=(y-b)/m=(z-c)/n. \qquad (P.U.\ 1939)$$

The general equation of the plane containing the given line is

$$A(x-a)+B(y-b)+C(z-c)=0, \qquad ...(i)$$

where A, B, C are any numbers subjected to the condition

$$Al+Bm+Cn=0, \qquad ...(ii)$$

The plane (i) will pass through (x_1, y_1, z_1), if

$$A(x_1-a)+B(y_1-b)+C(z_1-c)=0. \qquad .. (iii)$$

Eliminating A, B, C from (i), (ii) and (iii), we have

$$\begin{vmatrix} x-a, & y-b, & z-c \\ l, & m, & n \\ x_1-a, & y_1-b, & z_1-c \end{vmatrix}=0,$$

as the required equation.

3. Find the equation of the plane containing the line

$$\tfrac{1}{2}(x+2)=\tfrac{1}{3}(y+3)=-\tfrac{1}{4}(z-4)$$

and the point (0, 6, 0). [*Ans.* $3x+2y+6z-12=0$.

4. $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ and $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$,

are two straight lines. Find the equation of the plane containing the first line and parallel to the second. [*Ans.* $\Sigma(x-x_1)(m_1n_2-m_2n_1)=0$.

5. Show that the equation of the plane through the line

$$\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2} \text{ and parallel to } \frac{x-2}{2}=\frac{y-1}{-3}=\frac{z+4}{5}$$

is $26x-11y-17z-109=0$ and show that the point (2, 1, -4) lies on it. What is the geometrical relation between the two lines and the plane ?

6. Find the equation of the plane containing the line

$$-\tfrac{1}{3}(x+1)=\tfrac{1}{2}(y-3)=(z+2)$$

and the point (0, 7, -7) and show that the line $x=\frac{1}{3}(7-y)=\frac{1}{2}(z+7)$ lies in the same plane. [*Ans.* $x+y+z=0$.

7. Find the equation of the plane which contains the line

$$(x-1)/2=-y-1=(z-3)/4$$

and is perpendicular to the plane

$$x+2y+z=12.$$

Deduce the direction cosines of the projection of the given line on the given plane. (*L.U.*)

[*Ans.* $9x-2y-5z+4=0$; $4k, -7k, 10k$, where $k=1/\sqrt{(165)}$.

8. Find the equations, in the symmetrical form, of the projection of the line

$$\tfrac{1}{2}(x+1)=\tfrac{1}{3}(y+2)=\tfrac{1}{4}(z+3)$$

on the plane

$$x-2y+3z-4=0.$$

[*Ans.* $(x-\frac{1}{4})/10=(y+15/8)/29=(z-0)/16$.

3·4. Coplanar Lines. Condition for the coplanarity of lines. *To find the condition that the two given straight lines should intersect, i.e., be coplanar.*

Let the given straight lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \qquad \ldots(1)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}. \qquad \ldots(2)$$

If the lines intersect, they must lie in a plane. Equation of *any* plane containing the line (1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \qquad \ldots(i)$$

with the condition

$$Al_1+Bm_1+Cn_1=0. \qquad \ldots(ii)$$

The plane (*i*) will contain the line (2), if the point (x_2, y_2, z_2) lies upon it and the line is perpendicular to the normal to it. (§ 3·3). This requires

$$A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0, \qquad \ldots(iii)$$

$$Al_2+Bm_2+Cn_2=0. \qquad \ldots(iv)$$

Eliminating A, B, C from (*ii*), (*iii*), (*iv*), we get

$$\begin{vmatrix} x_2-x_1, & y_2-y_1, & z_2-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0, \qquad \ldots(A)$$

which is the required condition for the lines to intersect. Again eliminating A, B, C from (*i*), (*ii*), (*iv*) we get

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0,$$

which is the equation of the plane containing the two lines, in case they intersect.

Second Method. The condition for intersection may also be obtained as follows :—

$(l_1r_1+x_1, m_1r_1+y_1, n_1r_1+z_1)$ and $(l_2r_2+x_2, m_2r_2+y_2, n_2r_2+z_2)$ are the general co-ordinates of the points on the lines (1) and (2) respectively.

In case the lines intersect, these points should coincide for some values of r_1 and r_2. This requires

$$\begin{aligned}(x_1-x_2)+l_1r_1-l_2r_2&=0,\\ (y_1-y_2)+m_1r_1-m_2r_2&=0,\\ (z_1-z_2)+n_1r_1-n_2r_2&=0.\end{aligned}$$

Eliminating r_1, r_2, we have

$$\begin{vmatrix} x_1-x_2, & l_1, & l_2 \\ y_1-y_2, & m_1, & m_2 \\ z_1-z_2, & n_1, & n_2 \end{vmatrix}=0$$

which is the same condition as (A).

Note 1. In general, the equation

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0$$

represents the plane through (1) and parallel to (2), and the equation

$$\begin{vmatrix} x-x_2, & y-y_2, & z-z_2 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0$$

represents the plane through (2) and parallel to (1).

In case the lines are coplanar, the condition (A) shows that the point (x_2, y_2, z_2) lies on the first plane and the point (x_1, y_1, z_1) on the second. These two planes are then identical and contain both the intersecting lines.

Thus the equation of a plane containing two *intersecting lines* is obtained by finding the plane through one line and parallel to the other *or*, through one line and any point on the other.

Note 2. Two lines will intersect if, and only if, there exists a point whose co-ordinates satisfy the *four* equations, two of each line. But we know that three unknowns can be determined so as to satisfy three equations. Thus for intersection, we require that the four equations should be consistent among themselves, *i.e.*, the values of the unknowns x, y, z, as obtained from any three equations, should satisfy the fourth also. The condition of consistency of four equations containing three unknowns is obtained by eliminating the unknowns. It is sometimes comparatively more convenient to follow this method to obtain the condition of intersection or to prove the fact of intersection of two lines.

Note 3. The condition for the lines, whose equations, given in the unsymmetrical form, are

$$a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0\ ;$$
$$a_3x+b_3y+c_3z+d_3=0,\ a_4x+b_4y+c_4z+d_4=0\ ;$$

to be coplanar, *i.e.*, to intersect, as obtained by eliminating x, y, z from these equations, is

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = 0.$$

In case, this condition is satisfied, the co-ordinates of the point of intersection are obtained by solving any three of the four equations simultaneously.

Examples

1. *Prove that the lines*

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7},\quad \frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$$

intersect and find the co-ordinates of their point of intersection.

Now,

$$(r+4,\ -4r-3,\ 7r-1) \text{ and } (2r'+1,\ -3r'-1,\ 8r'-10)$$

are the general co-ordinates of points on the two lines respectively.

They will intersect if the three equations

$$r-2r'+3=0, \quad \ldots (i)$$
$$4r-3r'+2=0, \quad \ldots (ii)$$
$$7r-8r'+9=0, \quad \ldots (iii)$$

are simultaneously true.

(*i*) and (*ii*) give $r=1$, $r'=2$ which also, clearly, satisfy (*iii*). Hence the lines intersect and their point of intersection obtained by putting $r=1$, or $r'=2$ is $(5, -7, 6)$.

Note. This equation can also be solved by first finding the point satisfying three equations

$$\frac{x-4}{1}=\frac{y+3}{-4}; \quad \frac{y+3}{-4}=\frac{z+1}{7}; \quad \frac{x-1}{2}=\frac{y+1}{-3},$$

and then showing that the same point also satisfies the equation

$$\frac{y+1}{-3}=\frac{z+10}{8}.$$

2. *Show that the lines*

$$\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}, \quad \frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$$

are coplanar and find the equation of the plane containing them.

The equation of the plane containing the first line and parallel to the second is

$$\begin{vmatrix} x+3, & y+5, & z-7 \\ 2, & 3, & -3 \\ 4, & 5, & -1 \end{vmatrix}=0$$

or

$$6x-5y-z=0.$$

which is clearly satisfied by the point $(-1, -1, -1)$, a point on the second line. Hence this plane contains also the second line. Thus the two lines are coplanar and the equation of the plane containing them is

$$6x-5y-z=0.$$

3. *Show that the lines*

$$\frac{x+5}{3}=\frac{y+4}{1}=\frac{z-7}{-2},$$
$$3x+2y+z-2=0=x-3y+2z-13$$

are coplanar and find the equation to the plane in which they lie.

The *general* equation of the plane through the second line is

$$3x+2y+z-2+k(x-3y+2z-13)=0$$

or

$$3(3+k)+y(2-3k)+z(1+2k)-2-13k=0.$$

This will be parallel to the first line

if

$$3(3+k)+(2-3k)-2(1+2k)=0, \text{ } i.e., \text{ } k=\tfrac{9}{4}.$$

Hence the equation of the plane containing the second line and parallel to the first is

$$21x-19y+22z-125=0$$

which clearly passes through the point $(-5, -4, 7)$ and so contains also the first line.

Thus the two lines are coplanar and lie in the plane

$$21x-19y+22z-125=0.$$

Exercises

1. Show that the lines

$$\tfrac{1}{3}(x+4)=\tfrac{1}{5}(y+6)=-\tfrac{1}{2}(z-1)$$
$$3x-2y+z+5=0=2x+3y+4z-4$$

are coplanar. Find also the co-ordinates of their point of intersection and the equation of the plane in which they lie.

[*Ans.* $(2, 4, -3)$; $45x-17y+25z+53=0$.

2. Prove that the lines

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8} \; ; \; \frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$$

intersect. Find also their point of intersection and the plane through them.

[*Ans.* $(5, -7, 6)$; $11x=6y+5z+67$.

3. Prove that the lines

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7} \; ; \; \frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$$

intersect. Find their point of intersection and the plane in which they lie.

[*Ans.* $(1/2, -1/2, -3/2)$; $x-2y+z=0$.

4. Show that the lines

$$x+2y-5z+9=0=3x-y+2z-5 \; ;$$
$$2x+3y-z-3=0=4x-5y+z+3$$

are coplanar.

5. Prove that the lines

$$x-3y+2z+4=0=2x+y+4z+1 \; ;$$
$$3x+2y+5z-1=0=2y+z$$

intersect and find the co-ordinates of their point of intersection.

[*Ans.* $(3, 1, -2)$.

6. Prove that the lines

$$\frac{x-a}{a'}=\frac{y-b}{b'}=\frac{z-c}{c'} \text{ and } \frac{x-a'}{a}=\frac{y-b'}{b}=\frac{z-c'}{c}$$

intersect and find the co-ordinates of the point of intersection and the equation of the plane in which they lie. [*Ans.* $(a+a', b+b', c+c')$; $\Sigma x(bc'-b'c)=0$.

7. Show that the condition that the two straight lines

$$x=mz+a, \; y=nz+b, \text{ and } x=m'z+a', \; y=n'z+b'$$

should intersect is

$$(a-a')(n-n')=(b-b')(m-m').$$

8. Show that the plane which contains the two parallel lines

$$x-4=-\tfrac{1}{4}(y-3)=\tfrac{1}{5}(z-2), \; x-3=-\tfrac{1}{4}(y+2)=\tfrac{1}{5}z$$

is given by

$$11x-y-3z=35.$$

9. Find the equation of the plane passing through $x/l=y/m=z/n$, and perpendicular to the plane containing

$$x/m=y/n=z/l \text{ and } x/n=y/l=z/m. \qquad (D.\ U.\ Hons.\ 1949)$$

[*Ans.* $\Sigma(m-n)x=0$.

10. Show that the line $x+a=y+b=z+c$ intersect the four lines

(*i*) $x=0, y+z=3a$; (*ii*) $y=0, z+x=3b$; (*iii*) $z=0$; $x+y=3c$

(*iv*) $x+y+z=3k$, $a(a-k)^{-1}\,x+b(b-k)^{-1}\,y+c(c-k)^{-1}z=0$

at right angles if $a+b+c=0$.

11. Obtain the condition for the line

$$(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$$

to intersect the locus of the equations $ax^2+by^2=1$, $z=0$.

[*Ans.* $a(\alpha n-l\gamma)^2+b(\beta n-m\gamma)^2=n^2$.

3·5. Number of arbitrary constants in the equations of a straight line. *To show that there are four arbitrary constants in the equations of a straight line.*

A line PQ can be regarded as the intersection of *any* two planes through it. In particular, we may take the two planes perpendicular to two of the co-ordinate planes, say, YZ and ZX planes.

The equations of the planes through PQ perpendicular to the YZ and ZX planes are respectively of the forms

$$z=cy+d, \text{ and } z=ax+b$$

which are, therefore, the equations of the line PQ and contain four arbitrary constants a, b, c, d.

Hence the *equations of a straight line involve four arbitrary constants* as it is always possible to express them in the above form.

Note. The symmetrical form of the equations of a line apparently involves six constants x_1, y_1, z_1 ; l, m, n, but they are really equivalent to four arbitrary constants only as is shown below :

l, m, n, which are connected by the relation $l^2+m^2+n^2=1$ are equivalent to two independent constants only.

Also, of the three apparently independent numbers x_1, y_1, z_1, only two are independent as one of them can always be arbitrarily chosen as described below :—

A line cannot be parallel to all the co-ordinate planes. Let the given line, in particular, be not parallel to the YZ plane. If, now, x_1 be assigned any value, we may take the point where the line meets the plane $x=x_1$ at the point (x_1, y_1, z_1).

Hence we may give to x_1 any value we please. The three numbers x_1, y_1, z_1 are, therefore, equivalent to two independent constants only.

The fact that the general equations of a straight line contain four arbitrary constants may also be seen *directly* as follows :—

We see that

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}, \quad \frac{y-y_1}{m}=\frac{z-z_1}{n}$$

are equivalent to

$$x=\frac{l}{m}y+\frac{(mx_1-ly_1)}{m}, \quad y=\frac{m}{n}z+\frac{(ny_1-mz_1)}{n}$$

respectively, so that

$$\frac{l}{m}, \quad \frac{m}{n}, \quad \frac{mx_1-ly_1}{m}, \quad \frac{ny_1-mz_1}{n}$$

are the four arbitrary constants or parameters.

3·51. Determination of lines satisfying given conditions.

We now consider the various *sets of conditions* which determine a line.

We know that the equations of a straight line involve four arbitrary constants and hence any four geometrical conditions, each giving rise to one relation between the constants, fix a straight line.

It may be noted that the conditions for a line to intersect a given line or be perpendicular to it separately involve one relation between the constants and hence three more relations are required to fix the line.

A given condition may sometimes give rise to two relations between the constants as, for instance, the condition of the line

(*i*) to pass through a given point.

or (*ii*) to have a given direction.

In such cases only two more relations will be required to fix the straight line.

Equations of lines have already been discussed under the following sets of conditions :

(*i*) passing through a given point and having a given direction;

(*ii*) passing through two given points ;

(*iii*) passing through a point and parallel to two given planes ;

(*iv*) passing through a point and perpendicular to two given lines.

Some further sets of conditions which determine a line are given below :—

(*v*) passing through a given point and intersecting two given lines;

(*vi*) intersecting two given lines and having a given direction ;

(*vii*) intersecting a given line at right angles and passing through a given point;

(*viii*) intersecting two given lines at right angles ;

(*ix*) intersecting a given line parallel to a given line and passing through a given point ;

(*x*) passing through a given point and perpendicular to two given lines ;

and so on.

An Important Note : *If*

$$u_1=0=v_1 \text{ and } u_2=0=v_2,$$

be two straight lines, then the general equations of a straight line intersecting them both are

$$u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2,$$

where λ_1, λ_2 *are any two constant numbers.*

The line $u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2$ lies in the plane $u_1+\lambda_1 v_1=0$ which again contains the line $u_1=0=v_1$.

The two lines

$$u_1+\lambda_1 v_1=0=u_2+\lambda_2 v_2 \; ; \; u_1=0=v_1$$

are, therefore, coplanar and hence they intersect.

Similarly, the same line intersects the line $u_2=0=v_2$.
This conclusion will be found very helpful in what follows.
For the sake of illustration, we give below a few examples.

Examples

1. *Find the equations of the line that intersects the lines*

$$2x+y-4=0=y+2z\ ;\ x+3z=4,\ 2x+5z=8$$

and passes through the point $(2, -1, 1)$.

The line

$$2x+y-4+\lambda_1(y+2z)=0,\ x+3z-4+\lambda_2(2x+5z-8)=0$$

intersects the two given lines for all values of λ_1, λ_2.

This line will pass through the point $(2, -1, 1)$, if

$$-1+\lambda_1=0 \text{ and } 1+\lambda_2=0,$$

i.e., if $$\lambda_1=1,\ \lambda_2=-1.$$

The required equations, therefore, are

$$x+y+z=2 \text{ and } x+2z=4.$$

2. *Find the equations, to the line that intersects the lines*

$$2x+y-1=0=x-2y+3z\ ;$$
$$3x-y+z+2=0=4x+5y-2x-3$$

and is parallel to the line

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}.$$

The general equations of the lines intersecting the two given lines are

$$2x+y-1+\lambda_1(x-2y+3z)=0$$
$$3x-y+z+2+\lambda_2(4x+5y-2z-3)=0$$

which will be parallel to the given line if λ_1, λ_2 be so chosen that the two planes representing it are separately parallel to the given line.

This requires

$$(2+\lambda_1)+2(1-2\lambda_1)+3(3\lambda_1)=0,\ \textit{i.e.},\ \lambda_1=-\tfrac{2}{3}.$$

and $$(3+4\lambda_2)+2(-1+5\lambda_2)+3(1-2\lambda_2)=0,\ \textit{i.e.},\ \lambda_2=-\tfrac{1}{2}.$$

The required equations of the line, therefore, are

$$4x+7y-6z-3=0,\ 2x-7y+4z+7=0.$$

3. *A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equations*

$$x=y+a=z\ ;\ x+a=2y=2z\ ;$$

find out the co-ordinates of each of the points of intersection.

$P(r, r-a, r)$ and $P'(2r'-a, r', r')$ are the general co-ordinates of points on the two given lines

$$\frac{x}{1}=\frac{y+a}{1}=\frac{z}{1}\ ,\ \frac{x+a}{2}=\frac{y}{1}=\frac{z}{1}.$$

The direction cosines of PP' are proportional to

$$r-2r'+a,\ r-r'-a,\ r-r'.$$

Now, we choose r and r' such that the line PP' has direction cosines proportional to (2, 1, 2).

$$\therefore \qquad \frac{r-2r'+a}{2}=\frac{r-r'-a}{1}=\frac{r-r'}{2},$$

which give

$$r=3a,\ r'=a.$$

Putting $r=3a$ and $r'=a$ in the co-ordinates of P and P', we get

$$(3a,\ 2a,\ 3a) \text{ and } (a,\ a,\ a)$$

which are the required points of intersection.

4. *Find the equations of the perpendicular from the point* (3, −1, 11) *to the line*

$$\tfrac{1}{2}x=\tfrac{1}{3}(y-2)=\tfrac{1}{4}(z-3).$$

Obtain also the foot of the perpendicular.

The co-ordinates of any point on the given line are

$$2r,\ 3r+2,\ 4r+3.$$

This will be the required foot of the perpendicular if the line joining it to the point (3, −1, 11) be perpendicular to the given line. This requires

$$2(2r-3)+3(3r+2+1)+4(4r+3-11)=0 \text{ or } r=1.$$

Therefore the required foot is (2, 5, 7) and the required equations of the perpendiculars are

$$\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}.$$

Exercises

1. Find the equations of the perpendicular from

(*i*) (2, 4, −1) to $(x+5)=\tfrac{1}{4}(y+3)=-\tfrac{1}{9}(z-6)$,

(*ii*) (−2, 2, −3) to $(x-3)=\tfrac{1}{2}(y+1)=-\tfrac{1}{4}(z-2)$,

(*iii*) (0, 0, 0) to $x+2y+3z+4=0=2x+3y+4z+5$,

(*iv*) (−2, 2, −3) to $2x+y+z-7=0=4x+z-14$.

Obtain also the feet of the perpendiculars.

[*Ans.* (*i*) $\tfrac{1}{6}(x-2)=\tfrac{1}{3}(y-4)=\tfrac{1}{2}(z+1)$, (−4, 1, −3).
(*ii*) $\tfrac{1}{3}(x+2)=-(y-2)=(z+3)$, (4, 1, −2).
(*iii*) $-x/2=y=z/4$, (2/3, −1/3, −4/3).
(*iv*) $\tfrac{1}{6}(x+2)=-(y-2)=(z+3)$, (4, 1, −2).

2. A line with direction cosines proportional to (7, 4, −1) is drawn to intersect the lines

$$\frac{x-1}{3}=\frac{y-7}{-1}=\frac{z+2}{1},\quad \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-5}{4}.$$

Find the co-ordinates of the points of intersection and the length intercepted on it. [*Ans.* (7, 5, 0), (0, 1, 1), $\sqrt{(66)}$.

3. Find the equations to the line that intersects the lines

$$x+y+z=1,\ 2x-y-z=2\ ;\ x-y-z=3,\ 2x+4y-z=4$$

and passes through the point (1, 1, 1). Find also the points of intersection. (*P.U. 1939*)

[*Ans.* $(x-1)/0=(y-1)/1=(z-1)/3$; $(1, \tfrac{1}{2}, -\tfrac{1}{2})$; (1, 0, −2).

4. Find the equations to the straight lines drawn from the origin to intersect the lines

$$3x+2y+4z-5=0,\ 2x-3y+4z+1=0\ ;\ 2x-4y+z+6=0=3x-4y+z-3.$$

(P.U. 1942)

[*Ans.* $13x-13y+24z=0=8x-12y+3z$.

5. Obtain the equations of the line drawn through the point (1, 0, −1), and intersecting the lines

$$x=2y=2z\ ;\ 3x+4y=1\ ;\ 4x+5z=2.$$

[*Ans.* $-(x-1)/6=y=(z+1)/9$.

6. Find the equations to the line drawn parallel to $x/2=y/3=z/4$ so as to intersect the lines

$$9x+y+z+4=0=5x+y+3z\ ;\ x+2y-3z-3=0=2x-5y+3z+3.$$

[*Ans.* $(x+1)/2=y/3=z/4$.

7. Find the equations of the line drawn through the point (−4, 3, 1), parallel to the plane $x+2y-z=5$ so as to intersect the line

$$-(x+1)/3=(y-3)/2=-(z-2).$$

Find also the point of intersection.

[*Ans.* $(x+4)/3=-(y-3)=(z-1)$; (2, 1, 3).

8. Find the distance of the point (−2, 3, −4) from the line

$$(x+2)/3=(2y+3)/4=(3z+4)/5$$

measured parallel to the plane

$$4x+12y-3z+1=0.$$ [*Ans.* 17/2.

9. Find the equations of the straight line through the point (2, 3, 4) perpendicular to the X-axis and intersecting the line $x=y=z$.

[*Ans.* $x=2,\ 2y-z=2$.

10. Find the equations of the straight line through the origin which will intersect the lines

$$(x-1)/2=(y+3)/4=(z-5)/3,\ (x-4)/2=(y+3)/3=(z-14)/4$$

and prove that the secant is divided at the origin in the ratio 1 : 2.

11. Find the equations of the two lines through the origin which intersect the line $(x-3)/2=y-3=z$ at angles of 60°.

[*Ans.* $x=y/2=-z\ ;\ x=-y=z/2$.

12. The straight line which passes through the points (11, 11, 18), (2, −1, 3) is intersected by a straight line drawn through (15, 20, 8) at right angles to Z-axis ; show that the two lines intersect at the point (5, 3, 8).

13. A straight line is drawn through the origin meeting perpendicularly the straight line through (a, b, c) with direction cosines l, m, n ; prove that the direction cosines of the line are proportional to

$$a-lk,\ b-mk,\ c-nk \text{ where } k\equiv al+bm+cn.$$

14. From the point $P(a, b, c)$ perpendiculars PA, PB are drawn to the lines $y=2x$, $z=1$ and $y=-2x$, $z=-1$; find the co-ordinates of A and B.

Prove that, if P moves so that the angle APB is always a right angle, P always lies on the surface $12x^2-3y^2+25z^2=25$.

[*Ans.* $A[(2b+a)/5, (4b+2a)/5, 1]$; $B[(a-2b)/5, (4b-2a)/5, -1]$.

3·6. The shortest distance between two lines. *To show that the shortest distance between two lines lies along the line meeting them both at right angles.*

Let AB, CD be two given lines.

A line is completely determined if it intersects two lines at right angles. (*See* § 3·51. *Case viii*).

Thus, there is one and only one line which intersects the two given lines at right angles, say, at G and H.

GH is, then, the shortest distance between the two lines for, if, A, C be *any* two points, one on each of the two given lines, then GH is clearly the projection of AC on itself and, therefore,

$$GH = AC\cos\theta,$$

where θ is the angle between GH and AC.

Hence $GH < AC$.

Thus GH is the shortest distance between the two lines AB and CD.

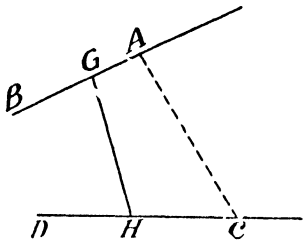

Fig. 16.

3·61. *To find the magnitude and the equations of the line of shortest distance between two straight lines.*

If AB, CD be two given lines and GH the line which meets them both at right angles at G and H, then GH is the line of shortest distance between the given lines and the length GH is the magnitude.

Let the equations of the given lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \qquad \ldots(i)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}, \qquad \ldots(ii)$$

and let the shortest distance GH lie along the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(iii)$$

Line (iii) is perpendicular to both the lines (i) and (ii). Therefore, we have

$$ll_1+mm_1+nn_1=0,$$
$$ll_2+mm_2+nn_2=0,$$

or
$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$
$$=\frac{1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}=\frac{1}{\sin\theta}$$

where θ is the angle between the given lines.

$$\therefore \quad l=\frac{m_1n_2-m_2n_1}{\sin\theta},\ m=\frac{n_1l_2-n_2l_1}{\sin\theta},\ n=\frac{l_1m_2-l_2m_1}{\sin\theta} \qquad \ldots(iv)$$

The line of shortest distance is perpendicular to both the lines. Therefore the magnitude of the shortest distance is the projection on the line of shortest distance of the line joining *any* two points, one on each of the given lines (i) and (ii).

Taking the projection of the join of (x_1, y_1, z_1), (x_2, y_2, z_2) on the line with direction cosines l, m, n, we see that the shortest distance

$$=(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n,$$

where l, m, n have the values as given in (iv).

To find the equations of the line of shortest distance, we observe that it is coplanar with both the given lines.

The equation of the plane containing the coplanar lines (*i*) and (*iii*) is

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l, & m, & n \end{vmatrix} = 0. \qquad \ldots(v)$$

and that of the plane containing the coplanar lines (*ii*) and (*iii*) is

$$\begin{vmatrix} x-x_2, & y-y_2, & z-z_2 \\ l_2, & m_2, & n_2 \\ l, & m, & n \end{vmatrix} = 0. \qquad \ldots(vi)$$

Thus (*v*) and (*vi*) are the two equations of the line of shortest distance, where l, m, n are given in (*iv*).

Note. Other methods of determining the shortest distance are given below where an example has been solved by three different methods.

Examples

1. *Find the magnitude and the equations of the line of shortest distance between the lines :*

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}, \qquad \ldots(i)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}. \qquad \ldots(ii)$$

First Method

Let l, m, n be the direction cosines of the line of shortest distance.

As it is perpendicular to the two lines, we have

$$3l-16m+7n=0,$$

and

$$3l+8m-5n=0.$$

$$\therefore \quad \frac{l}{24} = \frac{m}{36} = \frac{n}{72},$$

or

$$\frac{l}{2} = \frac{m}{3} = \frac{n}{6}.$$

Hence

$$l=\tfrac{2}{7},\ m=\tfrac{3}{7},\ n=\tfrac{6}{7}.$$

The magnitude of the shortest distance is the projection of the join of the points (8, −9, 10), (15, 29, 5), on the line of the shortest distance and is, therefore,

$$=7.\tfrac{2}{7}+38.\tfrac{3}{7}-5.\tfrac{6}{7}=14.$$

Again, the equation of the plane containing the first of the two

given lines and the line of shortest distance is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 2, & 3, & 6 \end{vmatrix} = 0,$$

or

$$117x+4y-41z-490=0.$$

Also the equation of the plane containing the second line and the shortest distance line is

$$\begin{vmatrix} x-15, & y-29, & z-5 \\ 3, & 8, & -5 \\ 2, & 3, & 6 \end{vmatrix} = 0,$$

or

$$9x-4y-z=14.$$

Hence the equations of the shortest distance line are

$$117x+4y-41z-490=0=9x-4y-z-14.$$

Second Method

$P(3r+8, -16r-9, 7r+10)$, $P'(3r'+15, 8r'+29, -5r'+5)$ are the general co-ordinates of the points on the two lines respectively. The direction cosines of PP' are proportional to

$$3r-3r'-7, \quad -16r-8r'-38, \quad 7r+5r'+5.$$

Now PP' will be the required line of shortest distance, if it is perpendicular to both the given lines, which requires

$$3(3r-3r'-7)-16(-16r-8r'-38)+7(7r+5r'+5)=0,$$

and $$3(3r-3r'-7)+8(-16r-8r'-38)-5(7r+5r'+5)=0.$$

or $$157r+77r'+311=0 \text{ and } 11r+7r'+25=0$$

which give $$r=-1, \; r'=-2.$$

Therefore co-ordinates of P and P' are

$$(5, 7, 3) \text{ and } (9, 13, 15).$$

Hence, the shortest distance $PP'=14$ and its equations are

$$\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}.$$

This method is sometimes very convenient and is specially useful when we require also the points where the line of shortest distance meets the two lines.

Third Method. This method depends upon the following considerations :—

Let AB, CD be the given lines and GH, the line of shortest distance between them.

Let 'α' denote the plane through AB and parallel to CD and let 'β' be the plane through CD and parallel to AB.

The line of shortest distance GH, being perpendicular to both AB, CD is normal to the two planes so that the two planes are parallel. The length GH of the shortest distance is, therefore, the distance between the parallel planes α and β. This distance between parallel planes being the distance of *any* point on one from the other, we see that it is enough to determine only one plane say 'α' and then the magnitude of the shortest distance is the distance of any point on the second line from the plane 'α'.

Again, we easily see that the plane through the lines AB, GH is perpendicular to the plane 'α' and the plane through CD, GH is perpendicular to the plane 'β' and, therefore, also to 'α'. Thus GH, *the line of shortest distance, is the line of intersection of the planes separately drawn through* AB, CD *perpendicular to the plane* 'α'.

We now solve the equation.

The equation of the plane containing the line (*i*) and parallel to the line (*ii*) is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 3, & 8, & -5 \end{vmatrix} = 0$$

or $$2x+3y+6z-49=0 \qquad \ldots(iii)$$

Perpendicular distance of the point (15, 29, 5), lying on the second line, from this plane

$$= \frac{30+87+30-49}{7}$$

$$= 14,$$

which is the required magnitude of the shortest distance.

The equation of the plane through (*i*) perpendicular to the plane (*iii*) is

$$\begin{vmatrix} x-8, & y+9, & z-10 \\ 3, & -16, & 7 \\ 2, & 3, & 6 \end{vmatrix} = 0$$

or $$117x+4y-41z-490=0. \qquad \ldots(iv)$$

The equation of the plane through (*ii*) and perpendicular to the plane (*iii*) is

$$\begin{vmatrix} x-15, & y-29, & z-5 \\ 3, & 8, & -5 \\ 2, & 3, & 6 \end{vmatrix} = 0$$

or $$9x-4y-z=14. \qquad \ldots(v)$$

Hence (*iv*), (*v*) are the equations of the line of shortest distance.

2. *Find the shortest distance between the axis of z and the line*

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0.$$

(D.U. Hons. 1948, B.U. 1955)

The *third method* given on page 56 will prove very convenient in this case.

Now, any plane through the second given line is

$$ax+by+cz+d+k(a'x+b'y+c'z+d')=0,$$

i.e., $$(a+ka')x+(b+kb')y+(c+kc')z+(d+kd')=0. \qquad ...(i)$$

It will be parallel to z-axis whose direction cosines are 0, 0, 1, if the normal to the plane is $\perp$ z-axis, *i.e.*, if,

$$0.(a+ka')+0.(b+kb')+1.(c+kc')=0,$$

i.e., $$k=-c/c'.$$

Substituting this value of k in (i), we see that the equation of the plane through the second line parallel to the first is

$$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0 \qquad ...(ii)$$

The required S.D. is the distance of *any* point on z-axis from the plane (ii).

$\therefore$ S.D. = perpendicular from (0, 0, 0), (a point on z-axis)

$$=\pm\frac{dc'-d'c}{\sqrt{[(ac'-a'c)^2+(bc'-b'c)^2]}}.$$

Exercises

1. Find the magnitude and the equations of the line of shortest distance between the two lines :

(*i*) $$\frac{x-3}{2}=\frac{y+15}{-7}=\frac{z-9}{5};\ \frac{x+1}{2}=\frac{y-1}{1}=\frac{z-9}{-3}.$$

(*ii*) $$\frac{x-3}{-1}=\frac{y-4}{2}=\frac{z+2}{1};\ \frac{x-1}{1}=\frac{y+7}{3}=\frac{z+2}{2}.$$

[*Ans.* (*i*) $x=y=z$; $4\sqrt{3}$.

(*ii*) $(x-4)=(y-2)/3=-(z+3)/5$; $\sqrt{35}$.

2. Find the length and the equations of the shortest distance line between

$$5x-y-z=0,\qquad x-2y+z+3=0\ ;$$
$$7x-4y-2z=0,\qquad x-y+z-3=0.$$

[***Hint.*** Transform the equations to the symmetrical form.]

[*Ans.* $17x+20y-19z-39=0=8x+5y-31z+67$; $13/\sqrt{75}$.

3. Find the magnitude and the position of the shortest distance between the lines

(*i*) $2x+y-z=0,\ x-y+2z=0\ ;\ x+2y-3z=4,\ 2x-3y+4z=5.$

(*ii*) $$\frac{x}{4}=\frac{y+1}{3}=\frac{z-2}{2}\ ;\ 5x-2y-3z+6=0,\ x-3y+2z-3=0.$$

[*Ans.* (*i*) $3x+z=0=22x-5y+4z-67$, $2\sqrt{14}/7$.

(*ii*) $7x-2y-11z+20=0=13x-13z+24$; $17\sqrt{6}/39$.

4. Obtain the co-ordinates of the points where the shortest distance between the lines

$$\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3},\ \frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$$

meets them. [*Ans.* (11, 11, 31) and (3, 5, 7).

5. Find the co-ordinates of the point on the join of $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes

$$3x-y-3z+32=0 \text{ and } 3x+2y-15z-8=0.$$

[Ans. $(-7, -1, -9)$.

6. Show that the shortest distance between the lines

$$x+a=2y=-12z \text{ and } x=y+2a=6z-6a$$

is $2a$.

7. Find the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}, \quad \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5};$$

show also that the lines are co-planar. *(P.U. 1926)*

8. Find the length and equations of the line of shortest distance between the lines

$$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2}, \quad \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}. \qquad \textit{(B.U. 1956)}$$

[Ans. 9 ; $32x+34y+13z-108=0$, $12x+33y+15z-81=0$.

9. Show that the length of the shortest distance between the line $z=x\tan\alpha$, $y=0$ and any tangent to the ellipse $x^2\sin^2\alpha+y^2=a^2$, $z=0$ is constant.

10. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes

$$y+z=0,\ z+x=0,\ x+y=0,\ x+y+z=a$$

is $2a/\sqrt{6}$ and that the three lines of shortest distance intersect at the point $x=y=z=-a$. *(D.U. Hons. 1960)*

3·7. Length of the perpendicular from a point to a line. *To find the length of the perpendicular from a given point $P(x_1, y_1, z_1)$ to a given line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

If H be the point (α, β, γ) on the given line and Q the foot of the perpendicular from P on it, we have,

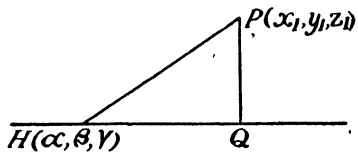

Fig. 17.

$$PQ^2=HP^2-HQ^2.$$

But $$HP^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2,$$

and $HQ=$ projection of HP on the given line

$$=l(x_1-\alpha)+m(y_1-\beta)+n(z-\gamma),$$

provided l, m, n, are the actual direction cosines.

$$\therefore \quad PQ^2=(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2 -[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2.$$

The expression for PQ^2 can be put in an elegant form as follows:

We have, by Lagrange's identity,

$$PQ^2=[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2][l^2+m^2+n^2] -[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]^2$$

$$=[l(y_1-\beta)-m(x_1-\alpha)]^2+[m(z_1-\gamma)-n(y_1-\beta)]^2 +[n(x_1-\alpha)-l(z_1-\gamma)]^2$$

$$=\begin{vmatrix} x_1-\alpha, & y_1-\beta \\ l, & m \end{vmatrix}^2+\begin{vmatrix} y_1-\beta, & z_1-\gamma \\ m, & n \end{vmatrix}^2+\begin{vmatrix} z_1-\gamma, & x_1-\alpha \\ n, & l \end{vmatrix}^2.$$

Exercises

1. Find the length of the perpendicular from the point $(4, -5, 3)$ to the line

$$\frac{x-5}{3}=\frac{y+2}{-4}=\frac{z-6}{5}.$$ [*Ans.* $\frac{\sqrt{(475)}}{5}$.

2. Find the locus of the point which moves so that its distance from the line $x=y=z$ is twice its distance from the plane $x+y+z=1$.

[*Ans.* $x^2+y^2+z^2+5xy+5yz+5zx-4x-4y-4z+2=0$.

3. Find the length of the perpendicular from the point $P(5, 4, -1)$ upon the line $\frac{1}{2}(x-1)=\frac{1}{3}y=\frac{1}{5}z$. [*Ans.* $\sqrt{(2109/110)}$.

3·8. Intersection of three planes. *To find the conditions that the three planes*

$$a_rx+b_ry+c_rz+d_r=0\ ;\quad (r=1, 2, 3)$$

should have a common line of intersection.

If these three planes have a common line of intersection, then

$$a_3x+b_3y+c_3z+d_3=0 \qquad \ldots(i)$$

must represent the same plane as

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0, \qquad \ldots(ii)$$

for some value of λ.

Comparing (i) and (ii), we get

$$\frac{a_1+\lambda a_2}{a_3}=\frac{b_1+\lambda b_2}{b_3}=\frac{c_1+\lambda c_2}{c_3}=\frac{d_1+\lambda d_2}{d_3}=k, \text{ (suppose)}$$

$$\therefore \quad \begin{aligned} a_1+\lambda a_2-ka_3&=0,\\ b_1+\lambda b_2-kb_3&=0,\\ c_1+\lambda c_2-kc_3&=0,\\ d_1+\lambda d_2-kd_3&=0. \end{aligned}$$

Eliminating λ and k from these four equations, taking them three by three, we obtain

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0, \quad \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0,$$

which are the required conditions.

Only *two* of these four conditions are *independent* for, if the planes have two points in common, they have the whole line in common and this fact requires only two conditions.

These four determinants will respectively be denoted by the letters $\triangle$, $\triangle_1$, $\triangle_2$, $\triangle_3$.

Note. The following is the Algebraic proof of the fact that only two of the above *four* conditions are independent, *i.e.*, if two of these four determinants vanish the other two must also vanish.

Let $$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0=\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}$$

$$\therefore \quad \begin{aligned} a_1A_1+a_2A_2+a_3A_3&=0\\ d_1A_1+d_2A_2+d_3A_3&=0 \end{aligned}$$

$$\therefore \quad \frac{A_1}{a_2d_3-a_3d_2}=\frac{A_2}{a_3d_1-a_1d_3}=\frac{A_3}{a_1d_2-a_2d_1}=\frac{1}{k}, \text{ (suppose)}$$

or $a_2d_3-a_3d_2=kA_1,\ a_3d_1-a_1d_3=kA_2,\ a_1d_2-a_2d_1=kA_3.$

Thus we obtain

$$b_1(a_2d_3-a_3d_2)+b_2(a_3d_1-a_1d_3)+b_3(a_1d_2-a_2d_1)$$
$$=k(A_1b_1+A_2b_2+A_3b_3)=0,$$

i.e.,
$$\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0.$$

Similarly it can be proved that

$$\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}=0.$$

Note. The same conditions will also be obtained in § 3·82 in a different manner.

3·81. Triangular prism. Def. *Three planes are said to form a triangular prism if the three lines of intersection of the three planes, taken in pairs, are parallel.*

Clearly, the three planes will form a triangular prism if the line of intersection of two of them be parallel to the third.

3·82. *To find the condition that the three planes*

$$a_rx+b_ry+c_rz+d_r=0\ ;\ (r=1,\ 2,\ 3)$$

should form a prism or intersect in a line.

The line of intersection of the first two planes is

$$\frac{x-(b_1d_2-b_2d_1)/(a_1b_2-a_2b_1)}{b_1c_2-b_2c_1}=\frac{y-(a_2d_1-a_1d_2)/(a_1b_2-a_2b_1)}{a_2c_1-a_1c_2}$$
$$=\frac{z}{a_1b_2-a_2b_1} \qquad \ldots(i)$$ (*See* Page 41)

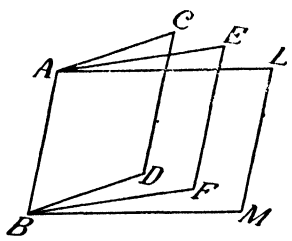

Fig. 18.

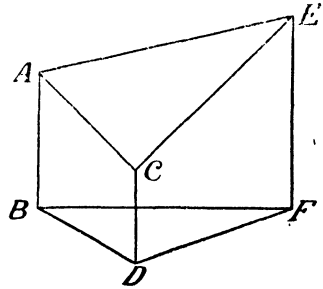

Fig. 19.

The three planes will form a triangular prism if this lines parallel to the third plane but does not lie in the same.

Then line (i) will be parallel to the third plane, if

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0,$$

or
$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} = 0,$$

i.e.,
$$\triangle = 0.$$

Again, the planes will intersect in a line if the line (i) lies in the plane $a_3x + b_3y + c_3z + d_3 = 0$. This requires :

(1) this line is parallel to the third plane which gives $\triangle = 0$, and

(2) the point $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right)$ lies on it which gives

$$a_3(b_1d_2 - b_2d_1) + b_3(a_2d_1 - a_1d_2) + d_3(a_1b_2 - a_2b_1) = 0,$$

or
$$\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix} = 0,$$

i.e.,
$$\triangle_3 = 0.$$

Thus the three planes will intersect in a line, if

$$\triangle = \triangle_3 = 0,$$

and will form a triangular prism, if

$$\triangle = 0 \text{ and } \triangle_3 \neq 0.$$

Note. Three distinct non-parallel planes behave in relation to each other in any of the following three ways :—

(i) They may intersect in a line which requires that two of the four determinants $\triangle$, $\triangle_1$, $\triangle_2$, $\triangle_3$ should vanish.

(ii) They may form a prism which requires that *only* $\triangle$ should vanish.

(iii) They may intersect in a unique finite point which requires that $\triangle \neq 0$.

Exercises

1. Show that the following sets of planes intersect in lines :

(i) $4x + 3y + 2z + 7 = 0$, $2x + y - 4z + 1 = 0$, $x - 7z - 2 = 0$.

(ii) $2x + y + z + 4 = 0$, $y - z + 4 = 0$, $3x + 2y + z + 8 = 0$.

2. Show that the following sets of planes form triangular prisms :

(i) $x + y + z + 3 = 0$, $3x + y - 2z + 2 = 0$, $2x + 4y + 7z - 7 = 0$.

(ii) $x - z - 1 = 0$, $x + y - 2z - 3 = 0$, $x - 2y + z - 3 = 0$.

3. Examine the nature of the intersection of the following sets of planes :

(i) $4x - 5y - 2z - 2 = 0$, $5x - 4y + 2z + 2 = 0$, $2x + 2y + 8z - 1 = 0$.

(ii) $2x + 3y - z - 2 = 0$, $3x + 3y + z - 4 = 0$, $x - y + 2z - 5 = 0$.

(iii) $5x + 3y + 7z - 4 = 0$, $3x + 26y + 2z - 9 = 0$, $7x + 2y + 10z - 5 = 0$.

(iv) $2x + 6y + 11 = 0$, $6x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$.

[*Ans.* (i) prism, (ii) point, (iii) line, (iv) prism.

4. Prove that the planes

$$x=cy+bz,\ y=az+cx,\ z=bx+ay,$$

pass through one line if

$$a^2+b^2+c^2+2abc=1,$$

and show that the line of intersection, then, is

$$\frac{x}{\sqrt{(1-a^2)}}=\frac{y}{\sqrt{(1-b^2)}}=\frac{z}{\sqrt{(1-c^2)}}. \qquad (B.U.)$$

5. Show that the planes

$$bx-ay=n,\ cy-bz=l,\ az-cx=m,$$

will intersect in a line if

$$al+bm+cn=0,$$

and the direction ratios of the line, then, are, a, b, c.

6. Prove that the three planes

$$bz-cy=b-c,\ cx-az=c-a,\ ay-bx=a-b,$$

pass through one line (*say* l), and the three planes

$$(c-a)z-(a-b)y=b+c,$$
$$(a-b)x-(b-c)z=c+a,$$
$$(b-c)y-(c-a)x=a+b,$$

pass through another line, say (l'). Show that the lines l and l' are at right angles to each other.

CHAPTER IV

INTERPRETATION OF EQUATIONS
LOCI

4·1. In Chapters II and III, it has been shown that any equation of the first degree in x, y, z represents a plane and two such equations together represent a straight line.

We now consider the nature of the geometrical loci represented by the equations of any degree.

4·2. Equation to a surface. *Locus of a variable point with its current co-ordinates x, y, z connected by a single equation $f(x, y, z)=0$ is a surface.*

Consider *any* point $(\alpha, \beta, 0)$ on the XY plane. The line through this point drawn parallel to the Z-axis, *viz.*, $x=\alpha$, $y=\beta$ meets the locus in points whose z-co-ordinates are given by the roots of the equation $f(\alpha, \beta, z)=0$.

As this equation has a finite number of roots, the number of points of the locus on *every* such line is also finite. Hence the locus, which is the assemblage of all such points for different values of α, β, must be a surface and not a solid.

Thus the equation $f(x, y, z)=0$ represents a surface.

4·21. Equations free from one variable. Cylinders. *Locus of the equation $f(x, y)=0$ is a cylinder with its generators parallel to Z axis.*

Consider the curve on the XY plane, whose two dimensional equation is $f(x, y)=0$. Let (α, β) be any point on it so that $f(\alpha, \beta)=0$.

Any point (α, β, z) on the line through this point, drawn parallel to OZ, therefore, satisfies the equation $f(x, y)=0$ and hence the whole line lies on its locus.

Thus the locus is the assemblage of lines, parallel to OZ drawn through the points, on the curve and is, therefore, a cylindrical surface.

Similarly the loci of the equations

$$f(y, z)=0 \text{ and } f(z, x)=0$$

are cylinders with generators parallel to the X-axis and the Y-axis respectively.

Ex. What surfaces are represented by the equations

(*i*) $x^2+y^2=a^2$, (*ii*) $x^2/a^2+y^2/b^2=1$, (*iii*) $y^2=4ax$.

(*iv*) $xy=c^2$, (*v*) $x^2/a^2-y^2/b^2=1$.

4·22. Equations containing only one variable. *Locus of the equation $f(x)=0$ is a system of planes parallel to the YZ plane.*

If $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots\ldots\ldots\ldots\ldots\alpha_n$ be the roots of the equation

$$f(x)=0,$$

then this equation is equivalent to

$$(x-\alpha_1)(x-\alpha_2).....(x-\alpha_n)=0$$

and, therefore, represents the planes

$$x=\alpha_1,\ x=\alpha_2,....., x=\alpha_n$$

which are parallel to the YZ plane.

Similarly the loci of the equations $f(y)=0$ and $f(z)=0$, are systems of planes respectively parallel to the ZX and XY planes.

4·3. Equations to a Curve. *Two equations*

$$f(x, y, z)=0,\ \phi(x, y, z)=0$$

together represent a curve.

The points, whose co-ordinates satisfy these equations simultaneously, are common to the two surfaces separately represented by them and, therefore, lie on their curve of intersection.

Hence the locus of a point whose current co-ordinates are connected by *two* equations is a curve.

Exercises

1. Find out the loci represented by

(*i*) $x^2/a^2+y^2/b^2=1, z=0$, (*ii*) $y^2=ax, z=c$,

(*iii*) $x^2+y^2=a^2, z^2=c^2$.

2. Show that the two curves

$$f(x, y, z)=0, \varphi(x, y, z)=0\ ;$$

$$f(x, y, z)-\lambda\varphi(x, y, z)=0, f(x, y, z)-\mu\varphi(x, y, z)=0$$

are identical.

3. Find the equations to the parabola whose focus is the point (1, 2, 3,), and directrix the line $x=y=z$.

[*Ans.* $x^2+y^2+z^2+2xy+2yz+2xz-6x-12y-18z+42=0=x-2y+z$.

4·4. Surfaces generated by straight lines. Ruled Surfaces. A straight line subjected to *three* conditions only, can take up an infinite number of positions. The locus of these lines is a surface called a *ruled* surface.

4·41. *To determine the ruled surface generated by a straight line intersecting three given lines*

$$u_1=0=v_1\ ;\ u_2=0=v_2\ ;\ u_3=0=v_3,$$

where $u_r\equiv a_rx+b_ry+c_rz+d_r,\ v_r\equiv a'_rx+b'_ry+c'_rz+d'_r$.

The straight line

$$u_1+\lambda_1v_1=0=u_2+\lambda_2v_2 \qquad \ldots (i)$$

intersects the first two lines for all values of λ_1, λ_2. (Note Page 50)

The condition of intersection of the line (*i*) with the third given line is a relation between λ_1, λ_2, say

$$f(\lambda_1, \lambda_2)=0. \qquad \ldots (ii)$$

The required ruled surface is, then, obtained by eliminating λ_1, λ_2 between (*i*) and (*ii*).

Another method will be indicated in the examples below.

4·42. Condition for the intersection of a straight line and a curve. If a straight line intersects a given curve, the co-ordinates of the points of intersection satisfy the four equations (two for the straight line and two for the curve) so that the four equations are simultaneously valid, *i.e.*, consistent. The condition for consistency is obtained by eliminating x, y, z from the four equations.

Examples

1. *Find the condition that the line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

should intersect the curve

$$xy=c^2,\ z=0. \qquad \ldots(ii)$$

Eliminating x, y, z from (i) and (ii), we obtain

$$\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)=c^2$$

which is the required condition.

2. *Find the locus of the line which intersects the three lines*

$$y=b,\ z=-c\ ;\ z=c,\ x=-a\ ;\ x=a,\ y=-b.$$

First Method. The line

$$y-b+\lambda_1(z+c)=0,\ z-c+\lambda_2(x+a)=0, \qquad \ldots(i)$$

which intersects the first two of the given lines, will also intersect the third,

if $$c-2a\lambda_2=\frac{2b}{\lambda_1}-c$$

or $$c=\frac{b}{\lambda_1}+a\lambda_2. \qquad \ldots(ii)$$

Eliminating λ_1, λ_2 from (i) and (ii), we obtain

$$c=-\frac{b(z+c)}{y-b}-a\frac{z-c}{x+a},$$

or $$c(x+a)(y-b)+a(z-c)(y-b)+b(x+a)(z+c)=0,$$

or $$ayz+bzx+cxy+abc=0,$$

which is the required locus.

Second Method.

Let the equations of the variable line intersecting the given lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

so that (α, β, γ) is *any* point on the line.

It will intersect the three given lines, if

$$\frac{b-\beta}{l}=-\frac{c+\gamma}{n}, \qquad \ldots(ii)$$

$$\frac{c-\gamma}{n} = -\frac{a+\alpha}{l}, \qquad \ldots(iii)$$

and

$$\frac{a-\alpha}{l} = -\frac{b+\beta}{m}, \qquad \ldots(iv)$$

(Note Page 49)

Eliminating l, m, n, between (ii), (iii) and (iv), we have

$$(a-\alpha)(b-\beta)(c-\gamma)+(a+\alpha)(b+\beta)(c+\gamma)=0.$$

As (α, β, γ) is any point on the variable line, the required locus is

$$(a-x)(b-y)(c-z)+(a+x)(b+y)(c+z)=0,$$

or

$$ayz+bzx+cxy+abc=0.$$

3. *Two skew lines are given by the equations*

$$ax+by=z+c=0 \; ; \; ax-by=z-c=0 \; ;$$

show that the lines which are perpendicular to the line with direction cosines proportional to l, m, n, and which meet the given lines generate the surface

$$abz(lx+my+nz)=c(a^2mx+b^2ly+abcn). \qquad (M.T.)$$

Let the variable line be

$$\frac{x-\alpha}{\lambda}=\frac{y-\beta}{\mu}=\frac{z-\gamma}{\nu}. \qquad \ldots(i)$$

This will be perpendicular to the line with direction cosines proportional to l, m, n,

if

$$l\lambda+m\mu+n\nu=0, \qquad \ldots(ii)$$

and will intersect the given lines

if

$$a\lambda(\gamma+c)+b\mu(\gamma+c)-\nu(a\alpha+b\beta)=0. \qquad \ldots(iii)$$

and

$$a\lambda(\gamma-c)-b\mu(\gamma-c)-\nu(a\alpha-b\beta)=0. \qquad \ldots(iv)$$

Eliminating λ, μ, ν from (ii), (iii), (iv), we have

$$\begin{vmatrix} a(\gamma-c), & -b(\gamma-c), & a\alpha-b\beta \\ a(\gamma+c), & b(\gamma+c), & a\alpha+b\beta \\ l, & m, & -n \end{vmatrix}=0,$$

or

$$l(ab\alpha\gamma-b^2c\beta)-m(ab\beta\gamma-a^2c\alpha)+nab(\gamma^2-c^2)=0.$$

The required locus, therefore, is

$$abz(lx+my+nz)=c(a^2mx+b^2ly+abcn).$$

4. *Find the locus of the line which moves parallel to the ZX plane and meets the curves*

$$xy=c^2, \; z=0 \; ; \; y^2=4cz, \; x=0 \; ;$$

verify that the locus contains the curves.

Let the variable line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad ...(i)$$

This will intersect the two given curves

if $$\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)=c^2 \qquad ...(ii)$$

and $$\left(\beta-\frac{\alpha m}{l}\right)^2=4c\left(\gamma-\frac{\alpha n}{l}\right). \qquad ...(iii)$$

The line (i) will be parallel to the ZX plane

if $$m=0. \qquad ..(iv)$$

Eliminating l, m, n from (ii), (iii) and (iv), we obtain

$$(c^2-\alpha\beta)(\beta^2-4c\gamma)=4c\alpha\beta\gamma,$$

so that $$y^2(c^2-xy)=4c^3z$$

is the required locus.

Putting x and z separately equal to zero in this equation we get $y^2=4cz$ and $xy=c^2$ and hence the verification.

5. *Find the equation of the surface traced out by lines which pass through a fixed point* (α, β, γ) *and intersect the curve*

$$ax^2+by^2=1,\ z=0.$$

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}; \qquad ...(i)$$

l, m, n, being variables.

It will intersect the given curve

if $$a\left(\alpha-\frac{l\gamma}{n}\right)^2+b\left(\beta-\frac{m\gamma}{n}\right)^2=1. \qquad ...(ii)$$

Eliminating l, m, n between (i) and (ii), we get

$$a\left(\alpha-\gamma\frac{x-\alpha}{z-\gamma}\right)^2+b\left(\beta-\gamma\frac{y-\beta}{z-\gamma}\right)^2=1,$$

or $$a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2,$$

which is the required equation to the surface.

Exercises

1. Prove that all lines which intersect the lines

$$y=mx,\ z=c\ ;\ y=-mx,\ z=-c$$

and are perpendicular to the X-axis lie on the surface

$$mxz=cy.$$

2. Find the locus of the lines which are parallel to the plane

$$x+y=0,$$

and which intersect the line $x-y=0=z$ and the curve

$$x^2=2az,\ y=0. \qquad [Ans.\ x^2-y^2=2az.$$

3. Find the surface generated by straight lines which intersect the lines $y=0, z=c\ ;\ x=0, z=-c\ ;$ and are parallel to the plane

$$lx+my+nz=0. \qquad [Ans.\ lx/(z+c)+my/(z-c)+n=0.$$

4. Show that the equation to the surface generated by straight lines intersecting the three lines

$$x=4a,\ y+2z=0\ ;\ x+4a=0,\ y=2z\ ;\ y=4a,\ x=2z,$$

is

$$x^2+y^2-4z^2=16a^2.$$

5. A variable line intersects the three lines

$$y-z=1,\ x=0\ ;\ z-x=1,\ y=0\ ;\ x-y=1,\ z=0.$$

Show that the locus is

$$x^2+y^2+z^2-2xy-2yz-2zx=1.$$

6. Obtain the locus of the straight line which intersects the circle

$$x^2+y^2=r^2,\ z=0$$

and the two straight lines $x=0=z+a\ ;\ y=0=z-a.$

$$[Ans.\ \ a^2[x^2(z-a)^2+y^2(z+a)^2]=r^2(z^2-a^2)^2.$$

7. Prove that the locus of a line which meets the lines

$$y=\pm mx,\ z=\pm c$$

and the circle

$$x^2+y^2=a^2,\ z=0$$

is $$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.$$

(D.U. Hons. 1948)

8. A straight line is drawn through a variable point on the ellipse $x^2/a^2+y^2/b^2=1,\ z=0$ to meet two fixed lines

$$y=mx,\ z=c\ ;\ y=-mx,\ z=-c.$$

Find the equation to the surface generated.

$$[Ans.\ \ a^2c^2m^2(cy-mxz)^2+b^2c^2(mcx-yz)^2=a^2bm^2(c^2-z^2)^2.$$

4·5. Equations of two skew lines in a simplified form. *To find the equations of two skew straight lines in a simplified form.*

Let the shortest distance between the two given lines AB and CD meet them at L and M and be of length $2c$.

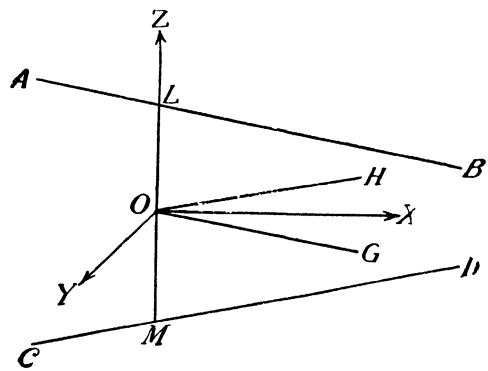

Fig. 20

Through O, the mid-point of LM, draw OG and OH parallel to AB and CD.

Take the bisectors of the angles between OG and OH as the X and Y-axis and LM as Z-axis. These three lines are mutually at right angles.

If the angle between the given lines be 2θ, the line OG makes angles θ, $\frac{1}{2}\pi-\theta$, $\frac{1}{2}\pi$ with the axes OX, OY, OZ so that the direction cosines of AB which is parallel to OG are

$$\cos\theta,\ \sin\theta,\ 0.$$

Also, since OH makes angles $-\theta$, $\frac{1}{2}\pi+\theta$, $\frac{1}{2}\pi$ with the axes, therefore the direction cosines of CD are

$$\cos\theta,\ -\sin\theta,\ 0.$$

Finally, the co-ordinates of L, M are

$$(0,\ 0,\ c) \text{ and } (0,\ 0,\ -c)$$

respectively, for $LM=2c$.

Thus the equations AB, CD are

$$\frac{x}{\cos\theta}=\frac{y}{\sin\theta}=\frac{z-c}{0}, \quad i.e.,\ y=x\tan\theta, \quad z=c;$$

and

$$\frac{x}{\cos\theta}=\frac{y}{-\sin\theta}=\frac{z+c}{0}, \quad i.e.,\ y=-x\tan\theta,\ z=-c$$

respectively.

Note 1. $(r,\ r\tan\theta,\ c)$ and $(p,\ -p\tan\theta,\ -c)$ are the general co-ordinates of points on the two lines ; r and p being the parameters.

Note 2. Solutions to certain problems relating to two non-intersecting straight lines are often simplified by taking the equations of the lines in the simplified form obtained above.

Exercises

1. *Find the surface generated by a straight line which meets two given skew lines at the same angle.*

Choosing the axes as in § 4·5, the equations of the two lines can be taken as

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \qquad \ldots(i)$$

and

$$\frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0} \qquad \ldots(ii)$$

so that the points $(r,\ mr,\ c)$ and $(p,\ -mp,\ -c)$ lie on these lines for all values of r and p.

The line joinng these points is

$$\frac{x-r}{r-p}=\frac{y-mr}{m(r+p)}=\frac{z-c}{2c}. \qquad \ldots(iii)$$

As it makes the same angle with both the lines (i) and (ii), we have

$$r-p+m^2(r+p)=r-p-m^2(r+p) \qquad \ldots(iv)$$

or

$$r+p=0.$$

From (iii) and (iv), we have

$$\frac{x-r}{2r}=\frac{z-c}{2c} \text{ and } y-mr=0,$$

so that eliminating r, we obtain

$$mcx=yz,$$

as the required locus.

2. A line intersects each of two fixed perpendicular non-intersecting lines so that the length intercepted is constant ; show that the locus of the middle point of the intercept is a circle.

3. A line of constant length has its extremities on two fixed straight lines ; find the locus of its middle point. *(D.U. Hons., 1959)*

4. Find the locus of a point which moves so that the perpendiculars drawn from it to two given skew lines are at right angles.

5. Two skew lines AP, BQ, inclined to one another at an angle of 60°, are intersected by the shortest distance between them at A, B, respectively, and

P, Q are points on the lines such that AQ is at right angles to BP ; prove that $AP.BQ=2AB^2$.

6. Two skew lines AP, BQ are met by the shortest distance between them at A, B and P, Q are points on them such that $AP=r$, $BQ=p$. If the planes APQ and BPQ are perpendiculars show that, pr, is constant.

7. AB and CD are two fixed skew lines. Planes are drawn through them at right angles to each other. Find the locus of their line of intersection. Show that the locus degenerates into two planes if AB is perpendicular to CD.

8. AB, CD are two perpendicular skew lines and the shortest distance between them meets the same at L and M ; O is the mid-point of LM ; P and P' are variable points on AB and CD such that $OP^2+OP'^2$ is constant. Find the locus of the line PP'.

9. Prove that the locus of the point which is equidistant from the lines

$$y-mx=0=z-c,\ y+mx=0=z+c$$

is the surface

$$mxy+(1+m^2)cz=0.$$

10. One edge of a tetrahedron is fixed in magnitude as well as position, and the opposite edge is of given length and lies along a fixed straight line. Show that the locus of the centroid of the tetrahedron is a straight line.

11. The length of two opposite edges of a tetrahedron are a, b ; the shortest distance between them is $2c$ and the angle between them is α ; prove that its volume is $(abc \sin \alpha)/3$.

12. A, B, C and A', B', C' are two sets of points on two skew lines. Prove that if

$$AB : BC=A'B' : B'C',$$

the middle points of AA', BB', CC' are collinear. (*M.T.*)

13. Lines are drawn to intersect the lines

$$y-mx=0=z-c \text{ and } y+mx=0=z+c$$

and to make a constant angle with z-axis. Show that the locus of their mid-points is an ellipse whose eccentricity is

$$(1-m^4)^{\frac{1}{2}} \text{ or } (m^4-1)^{\frac{1}{2}} \Big/ m^2$$

according as $m^2<1$ or >1.

14. AA' is the common perpendicular of two skew lines PQA, $P'Q'A'$; P, Q being any two points on the first line and P', Q' any two points on the second. Prove that the common perpendicular of AA' and the line joining the mid-points of PP', QQ' bisects AA'. (*M.T.*)

CHAPTER V

TRANSFORMATION OF CO-ORDINATES

5·1. The co-ordinates of a point in space are always determined *relatively* to any assigned system of axes, generally called the *frame of reference* and they change with the change in the frame of reference. We shall now obtain the formulae connecting the co-ordinates of a point relative to two different frames of reference.

5·11. Change of origin. *To change the origin of co-ordinates without changing the directions of axes.*

Let OX, OY, OZ, be the original axes and $O'X'$, $O'Y'$, $O'Z'$, the new axes respectively parallel to the original axes. Let the co-ordinates of O' referred to the original axes be (f, g, h).

Let the co-ordinates of any point P be x, y, z and x', y', z' referred to the original and the new axes respectively.

Draw PL perpendicular to the parallel planes YOZ and $Y'O'Z'$ meeting them at L and L' so that

$$LP=x \text{ and } L'P=x'.$$

Now, LL' is equal to the length of the perpendicular from O' to the YOZ plane and is, therefore $=f$.

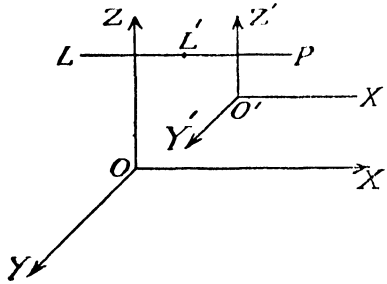

Fig. 21.

Also $$LP=LL'+L'P$$

$\therefore$ $$\mathbf{x=x'+f.}$$

Similarly

$$\mathbf{y=y'+g}, \text{ and } \mathbf{z=z'+h}.$$

Hence, if in the equation to any surface, we change

$$x, y, z$$

to

$$x+f, y+g, z+h$$

respectively, we obtain the equation to the same surface referred to the point (f, g, h) *as origin.*

Ex. Find the equations of the plane $2x+3y+4z=7$ referred to the point $(2, -3, 4)$ as origin ; directions of the axes remaining the same.

[*Ans.* $2x+3y+4z+4=0$.

5·12. Change of the directions of axes. *To change the directions of axes without changing the origin.*

Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the respective direction cosines of the new axes OX', OY', OZ' referred to the original axes OX, OY, OZ.

Let x, y, z and x', y', z' be the co-ordinates of any point P referred to the two systems of axes.

Draw $PN \perp X'OY'$ plane meeting it in N' and also $N'L' \perp OX'$ meeting it in L' so that

$OL'=x'$; $L'N'=y'$; $N'P=z'$.

Now, the projection of OP being equal to the sum of the projections of OL', $L'N'$, $N'P$ on OX', we have,

$$\left.\begin{aligned} \mathbf{x}&=\mathbf{l_1x'}+\mathbf{l_2y'}+\mathbf{l_3z'}\ ,\\ \text{Similarly } \mathbf{y}&=\mathbf{m_1x'}+\mathbf{m_2y'}+\mathbf{m_3z'},\\ \text{and } \mathbf{z}&=\mathbf{n_1x'}+\mathbf{n_2y'}+\mathbf{n_3z'}. \end{aligned}\right\} \ldots(\text{A})$$

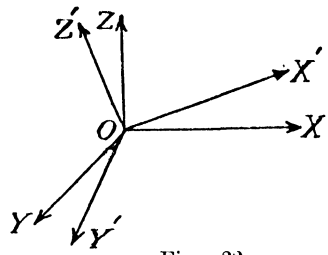

Fig. 22.

By a method similar to the one adopted, we can show that

$$\left.\begin{aligned} \mathbf{x'}&=\mathbf{l_1x}+\mathbf{m_1y}+\mathbf{n_1z}\ ;\\ \mathbf{y'}&=\mathbf{l_2x}+\mathbf{m_2y}+\mathbf{n_2z}\ ;\\ \mathbf{z'}&=\mathbf{l_3x}+\mathbf{m_3y}+\mathbf{n_3z}\ ; \end{aligned}\right\} \ldots(\text{B})$$

The results (A) and (B) can easily be written down with the help of the following table :

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Exercises

1. Find the equation of the surface

$$3x^2+5y^2+3z^2+2yz+2zx+2xy=1$$

with reference to axes through the same origin and with direction cosines proportional to $(-1, 0, 1)$, $(1, -1, 1)$, $(1, 2, 1)$. [*Ans.* $2x^2+3y^2+6z^2=1$.

2. Show that the equation $lx+my+nz=0$ becomes $z=0$, when referred to new axes through the same origin with direction cosines

$$\frac{-m}{\sqrt{(l^2+m^2)}}, \frac{l}{\sqrt{(l^2+m^2)}}, 0\ ; \frac{-ln}{\sqrt{(l^2+m^2)}}, \frac{-mn}{\sqrt{(l^2+m^2)}}, \sqrt{(l^2+m^2)}\ ;\ l, m, n.$$

Hence show that the curve $ax^2+by^2=2z$, $lx+my+nz=0$ is a rectangular hyperbola if $(a+b)n^2+am^2+bl^2=0$.

5·13. *The degree of a surface is unaltered by any transformation of axes.*

Since, for x, y, z we always put expressions of the first degree in x, y, z, the degree cannot increase.

Also, it cannot decrease for, otherwise, on retransforming it must increase.

5·2. Relations between the direction cosines of three mutually perpendicular lines.

l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 being the direction cosines of three mutually perpendicular lines OX, OY, OZ, we have the relations

$$\left.\begin{aligned} l_1^2+m_1^2+n_1^2&=1 \\ l_2^2+m_2^2+n_2^2&=1 \\ l_3^2+m_3^2+n_3^2&=1 \end{aligned}\right\} \quad \ldots(A)$$

$$\left.\begin{aligned} l_1l_2+m_1m_2+n_1n_2&=0 \\ l_2l_3+m_2m_3+n_2n_3&=0 \\ \text{and} \quad l_3l_1+m_3m_1+n_3n_1&=0 \end{aligned}\right\} \quad \ldots(B)$$

(Cor. 3, Page 17)

Thus these six relations exist between nine direction cosines. They can also be expressed in another form as shown below.

Now, l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3 are clearly the direction cosines, of the original axes OX, OY, OZ referred to the new. Therefore, we have the relations

$$\left.\begin{aligned} l_1^2+l_2^2+l_3^2&=1 \\ m_1^2+m_2^2+m_3^2&=1 \\ n_1^2+n_2^2+n_3^2&=1 \end{aligned}\right\} \quad \ldots(C)$$

and

$$\left.\begin{aligned} l_1m_1+l_2m_2+l_3m_3&=0 \\ m_1n_1+m_2n_2+m_3n_3&=0 \\ n_1l_1+n_2l_2+n_3l_3&=0. \end{aligned}\right\} \quad \ldots(D)$$

The relations A, B, C, D are not independent.

In fact the relations C, D can be algebraically deduced from the relations A, B and *vice versa*, without any geometrical considerations at all.

Cor. *If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of three mutually perpendicular straight lines, then*

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = \pm 1.$$

For, if D be the given determinant, we have

$$D^2 = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2+m_1^2+n_1^2, & l_1l_2+m_1m_2+n_1n_2, & l_1l_3+m_1m_3+n_1n_3 \\ l_1l_2+m_1m_2+n_1n_2, & l_2^2+m_2^2+n_2^2, & l_2l_3+m_2m_3+n_2n_3 \\ l_1l_3+m_1m_3+n_1n_3, & l_3l_2+m_3m_2+n_3n_2, & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} = 1.$$

Hence $D = \pm 1$.

5·3. Invariants. *If, by any change of rectangular axes without change of origin, the expression*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

becomes

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy$$

then

(*i*) $a+b+c=a'+b'+c'$,

(*ii*) $ab+bc+ca-f^2-g^2-h^2=a'b'+b'c'+c'a'-f'^2-g'^2-h'^2$,

(*iii*) $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}.$$

Consider two sets of rectangular axes

$$Ox, Oy, Oz\ ;\ OX, OY, OZ$$

through the same origin O. Let P be any point so that if (x, y, z), (X, Y, Z) be the co-ordinates of the same relative to the two systems of axes, we have

$$x^2+y^2+z^2=OP^2=X^2+Y^2+Z^2.$$

Thus we see that

$$x^2+y^2+z^2,$$

becomes

$$X^2+Y^2+Z^2.$$

Also, as given,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$

becomes

$$a'X^2+b'Y^2+c'Z^2+2f'YZ+2g'ZX+2h'XY.$$

Then if λ be any constant number, the expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+\lambda(x^2+y^2+z^2)$$
$$=(a+\lambda)x^2+(b+\lambda)y^2+(c+\lambda)z^2+2fyz+2gzx+2hxy \quad \ldots(1)$$

becomes

$$a'X^2+b'Y^2+c'Z^2+2f'YZ+2g'ZX+2h'XY+\lambda(X^2+Y^2+Z^2)$$
$$=(a'+\lambda)X^2+(b'+\lambda)Y^2+(c'+\lambda)Z^2+2f'YZ+2g'ZX+2h'XY \quad \ldots(2)$$

If now, for any value of λ, the expression

(1) becomes a product of two linear factors, then, for the same value of λ, the expression

(2) must also become a product of two linear factors. This follows from the fact that the degree of an expression does not change as a result of the change of axes so that the linear factors of (1) will become the two linear factors of (2).

Now, by § 2·8, P. 37, the values of λ for which the expression (1) and (2) are the products of linear factors are respectively the roots of the cubic equations

$$\begin{vmatrix} a+\lambda & h & g \\ h & b+\lambda & f \\ g & f & c+\lambda \end{vmatrix} = 0, \quad \begin{vmatrix} a'+\lambda & h' & g' \\ h' & b'+\lambda & f' \\ g' & f' & c'+\lambda \end{vmatrix} = 0$$

i.e., $\lambda^3+\lambda^2(a+b+c)+\lambda(bc+ca+ab-f^2-g^2-h^2)+D=0, \quad \ldots(3)$

$\lambda^3+\lambda^2(a'+b'+c')+\lambda(b'c'+c'a'+a'b'-f'^2-g'^2-h'^2)+D'=0, \quad \ldots(4)$

where

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \qquad D'=\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

As the equations (3) and (4) have the same roots, we see that

$$\frac{1}{1}=\frac{a+b+c}{a'+b'+c'}=\frac{bc+ca+ab-f^2-g^2-h^2}{b'c'+c'a'+a'b'-f'^2-g'^2-h'^2}=\frac{D}{D'},$$

so that

$$a+b+c=a'+b'+c',$$

$$bc+ca+ab-f^2-g^2-h^2=b'c'+c'a'+a'b'-f'^2-g'^2-h'^2,$$

$$D=D'.$$

Note 1. The result obtained above shows that if in relation to any second degree homogeneous expression

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy,$$

x, y, z be subjected to any change of rectangular axes without change of origin, then

$$a+b+c,\ bc+ca+ab-f^2-g^2-h^2,\ D$$

are *invariants.*

Note 2. It may be seen that

$$bc+ca+ab-f^2-g^2-h^2=A+B+C,$$

where A, B, C are the co-factors of a, b, c in the determinant D.

Ex. Show directly by changing

$$x, y, z, \text{ to } x+p,\ y+q,\ z+r$$

respectively that

$$a+b+c,\ A+B+C,\ D$$

are also invariants for change of origin.

[In fact, as may easily be seen, the co-efficients a, b, c, f, g, h, are themselves separately invariants for a change of origin without change in the direction of axes].

Ex. 1. *OA, OB, OC are three mutually perpendicular lines through the origin, and their direction cosines are*

$$l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3.$$

If $OA=OB=OC=a$, prove that the equation to the plane ABC is

$$(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)z=a.$$

Let the required equation be

$$lx+my+nz+p=0. \qquad ...(i)$$

The co-ordinates of A are (al_1, am_1, an_1).

The plane (i) passes through A. Therefore, we have

$$a(ll_1+mm_1+nn_1)+p=0. \qquad ...(ii)$$

Similarly, we have

$$a(ll_2+mm_2+nn_2)+p=0, \qquad .. (iii)$$

$$a(ll_3+mm_3+nn_3)+p=0. \qquad ...(iv)$$

Multiplying (ii), (iii), (iv) by l_1, l_2, l_3 respectively, and adding, we get

$$al+p(l_1+l_2+l_3)=0,$$

(From relations D. Page 74)

or
$$\frac{l}{p}=-\frac{l_1+l_2+l_3}{a}.$$

Similarly

$$\frac{m}{p}=-\frac{m_1+m_2+m_3}{a},$$

and
$$\frac{n}{p}=-\frac{n_1+n_2+n_3}{a}.$$

Making substitutions, in (i), we get the required result.

Ex. 2. l_r, m_r, n_r ; $(r=1, 2, 3)$ are direction cosines of three mutually perpendicular straight lines and

$$\frac{a}{l_1}+\frac{b}{m_1}+\frac{c}{n_1}=0, \quad \frac{a}{l_2}+\frac{b}{m_2}+\frac{c}{n_2}=0.$$

Prove that

$$a/l_3+b/m_3+c/n_3=0 \text{ and } a : b : c=l\ l_2l_3 : m_1m_2m_3 : n_1n_2n_3.$$

Ex. 3. If three rectangular axes be rotated about the line given by

$$x/l=y/m=z/n$$

into new positions and the direction cosines of the new axes referred to the old are l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 ; and, if

$$l_1=+(m_2n_3-m_3n_2)$$

then
$$l(m_3\pm n_2)=m(n_1+l_3)=n(l_2+m_1). \qquad (B.U.)$$

Examples

1. *Show that the planes*

$$3x-6y-5z+3=0, \qquad ...(i)$$

$$6x-9y-8z+3=0, \qquad ...(ii)$$

$$x-y-z+2=0. \qquad ...(iii)$$

form a triangular prism. Find the area and the lengths of the edges of its normal section.

Symmetrical form of the equations of the lines of intersection of the first two planes is

$$\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z}{3},$$

and, as may be easily shown, this line is parallel to the third plane but does not wholly lie in it. Hence the planes form a prism.

Normal sections of the prism are congruent triangles.

We consider the normal section through the origin. Equation of the plane of this section is

$$x-2y+3z=0. \qquad \ldots(iv)$$

Co-ordinates of the three vertices of the triangular section are obtained by solving simultaneously each of three pairs of the given equations with the equation (iv).

Thus the vertices are

$$A(-\tfrac{41}{14}, -\tfrac{16}{14}, -\tfrac{3}{14}),\ B(-\tfrac{71}{14}, -\tfrac{40}{14}, -\tfrac{3}{14}),\ C(\tfrac{15}{14}, \tfrac{12}{14}, -\tfrac{3}{14}).$$

Therefore, the lengths of the edges AB, BC, CA are

$$\frac{\sqrt{1512}}{14},\quad \frac{\sqrt{10136}}{14},\quad \frac{\sqrt{3920}}{14}.$$

Let $\triangle$ be the area of $\triangle ABC$. The co-ordinates of the projections $A'B'C'$ of A, B, C, on the XY plane are

$$(-\tfrac{41}{14}, -\tfrac{16}{14}, 0)\ ;\ (-\tfrac{71}{14}, -\tfrac{40}{14}, 0)\ ;\ (\tfrac{15}{14}, \tfrac{12}{14}, 0)$$

Let $\triangle_z$ be the area of $\triangle A'B'C'$. Therefore,

$$\triangle_z=\tfrac{1}{2}\begin{vmatrix} -\frac{41}{14}, & -\frac{16}{14}, & 1 \\ -\frac{71}{14}, & -\frac{40}{14}, & 1 \\ \frac{15}{14}, & \frac{12}{14}, & 1 \end{vmatrix}=\tfrac{9}{7}$$

Let θ be the angle between the plane (iv) and XOY plane. Therefore,

$$\cos\theta=\frac{3}{\sqrt{(14)}}$$

Also $\triangle_z=\triangle\cos\theta.$

$$\therefore \quad \triangle=\frac{\triangle_z}{\cos\theta}=\frac{9}{7}\cdot\frac{\sqrt{14}}{3}=\frac{3}{7}\sqrt{14}.$$

2. *Find the equations of the line of the greatest slope on the plane*

$$3x-4y+5z-5=0$$

drawn through the point $(3, -4, -4)$; *given that the plane*

$$4x-5y+6z-6=0$$

is horizontal.

Line of greatest slope on a given plane, drawn through a given point on the plane, is the line through the point perpendicular to the line of intersection of the given plane with any horizontal plane.

We have, thus, to find the line through $A(3, -4, -4)$ perpendicular to the line of intersection of the planes

$$3x-4y+5z-5=0,$$
$$4x-5y+6z-6=0.$$

Equations of this line in the symmetrical form are

$$\frac{x+1}{1}=\frac{y+2}{2}=\frac{z}{1}, \qquad \ldots(i)$$

so that the general co-ordinates of any point P on the line are

$$r-1,\ 2r-2,\ r.$$

The line AP will be perpendicular to (i), if

$$1(r-4)+2(2r+2)+1(r+4)=0,\ i.e.,\ r=-\tfrac{2}{3}$$

Thus, the co-ordinates of P are

$$(-\tfrac{5}{3},\ -\tfrac{10}{3},\ -\tfrac{2}{3}).$$

Hence the line, AP, of greatest slope is

$$\frac{x-3}{-7}=\frac{y+4}{1}=\frac{z+4}{5}.$$

3. *CP, CQ, are conjugate diameters of the ellipse*

$$x^2/a^2+y^2/b^2=1,\ z=-c\ ;$$

$C'P'$, $C'Q'$ are conjugate diameters of the ellipse

$$x^2/a^2+y^2/b^2=1,\ z=-c\ ;$$

drawn in the same direction as CP and CQ. Find the locus of the lines PQ' or $P'Q$.

Let P be $(a\cos\theta,\ b\sin\theta,\ c)$. Therefore, Q, P',Q' are

$$(-a\sin\theta,\ b\cos\theta,\ c),\ (a\cos\theta,\ b\sin\theta,\ -c)$$
$$(-a\sin\theta,\ b\cos\theta,\ -c)$$

respectively.

Equations of PQ' are

$$\frac{x-a\cos\theta}{a(\cos\theta+\sin\theta)}=\frac{y-b\sin\theta}{b(\sin\theta-\cos\theta)}=\frac{z-c}{2c} \qquad \ldots(i)$$

The locus will be obtained on eliminating θ from the equations (i)

The equations (i) can be written as

$$\frac{x}{a}=\frac{z+c}{2c}\cos\theta+\frac{z-c}{2c}\sin\theta,$$

$$\frac{y}{b}=-\frac{z-c}{2c}\cos\theta+\frac{z+c}{2c}\sin\theta.$$

Squaring and adding, we obtain

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{z+c}{2c}\right)^2+\left(\frac{z-c}{2c}\right)^2$$

or

$$\frac{2x^2}{a^2}+\frac{2y^2}{b^2}-\frac{z^2}{c^2}=1,$$

as the required locus.

It may be shown that the locus of $P'Q$ is the same surface.

4. *Show that the equations of the planes through the lines which bisect the angles between the lines*

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}\ \text{and}\ \frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$$

and perpendicular to the plane containing them, are

$$(l \pm l')x+(m \pm m')y+(n \pm n')z=0. \qquad (P.U.\ 1945)$$

Let OA, OB be the given lines. Take points A and B on the lines such that

$$OA=OB=r.$$

Take another point A' on the line OA produced such that O is the mid-point of AA'.

The co-ordinates of A, B, A' are

$$(lr, mr, nr), (l'r, m'r, n'r), (-lr, -mr, -nr)$$

respectively.

Let P, Q, be the mid-points of AB and $A'B$ respectively so that OP, OQ are the bisectors of the angles between OA and OB. We have

$$P \equiv [\tfrac{1}{2}(l+l')r, \tfrac{1}{2}(m+m')r, \tfrac{1}{2}(n+n')r].$$
$$Q \equiv [\tfrac{1}{2}(l'-l)r, \tfrac{1}{2}(m'-m)r, \tfrac{1}{2}(n'-n)r].$$

Thus the lines OP, OQ are

$$\frac{x}{l+l'}=\frac{y}{m+m'}=\frac{z}{n+n'}, \quad \frac{x}{l'-l}=\frac{y}{m'-m}=\frac{z}{n'-n} \qquad \ldots(2)$$

The lines OA, OB, OP, PQ are all coplanar.

Let OR be normal to this plane.

The lines OP, OQ and OR are mutually perpendicular so that the planes POQ (*i.e.*, the plane AOB), QOR, ROP are also mutually perpendicular.

The plane QOR passes through a bisector OQ and is perpendicular to the plane AOB so that it is one of the required planes. Being perpendicular to the line OP, its equation is

$$(l+l')x+(m+m')y+(n+n')z=0.$$

Similarly POR is the other required plane. Being perpendicular to the line OQ, its equation is

$$(l-l')x+(m-m')y+(n-n')z=0.$$

Revision Exercises I

1. Find the volume of the tetrahedron formed by the planes

$$lx+my+nz=p,\ lx+my=0,\ my+nz=0,\ nz+lx=0.$$

(D. U. Hons., 1948)

[*Ans.* $2p^3/3lmn$

2. Show that the straight lines

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}, \quad \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}, \quad \frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

will lie in one plane, if

$$l(b-c)/\alpha+m(c-a)/\beta+n(a-b)/\gamma=0. \qquad (P.U.\ 1942)$$

[The three lines have a point in common, *viz.*, the origin. They will be coplanar, if there exists a line through the origin perpendicular to each of them. If λ, μ, ν be the direction cosines of this line, we have

$$\lambda\alpha+\mu\beta+\nu\gamma=0,\ a\alpha\lambda+b\beta\mu+c\gamma\nu=0,\ l\lambda+m\mu+n\nu=0.$$

Eliminating λ, μ, ν we have the given condition.]

3. Show that the lines

$$\frac{x}{\alpha/a}=\frac{y}{\beta/b}=\frac{z}{\gamma/c}\ ;\ \frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}\ ;\ \frac{x}{\alpha a}=\frac{y}{\beta b}=\frac{z}{\gamma c}$$

are coplanar if $a=b$ or $b=c$ or $c=a$.

4. Show that the triangle whose vertices have co-ordinates (a, b, c), (b, c, a) and (c, a, b) is an equilateral triangle. Find the co-ordinates of the vertices of the two regular tetrahedra described on the above equilateral triangle as base.

(C.U. 1915) [*Ans.* (f, f, f) where $f=\dfrac{\Sigma a+2\sqrt{(\Sigma a^2-\Sigma bc)}}{3}$.

5. If two opposite edges of a tetrahedron are equal in length and are at right angles to the line joining their middle points, show that the other two pairs of opposite edges have the same property.

6. Two edges, AB, CD of a tetrahedron $ABCD$ are perpendicular ; show that the distance between the mid-points of AD and BC is equal to the distance between the mid-points of AC and BD.

7. Planes are drawn so as to make an angle of 60° with the line $x=y=z$ and an angle of 45° with the line $x=0=y-z$. Show that all these planes make an angle of 60° with the plane $x=0$.

Find the equations of the planes of this family which are 3 units distant from the point (2, 1, 1). [*Ans.* $2x+(2\pm\sqrt{2})\,y+(2\mp\sqrt{2})\,z=20$ or -4.

8. A plane meets a set of three mutually perpendicular planes in the sides of a triangle whose angles are A, B, C. Show that the first plane makes with the other three planes angles, the squares of whose cosines are

$$\cot B\cot C,\ \cot C\cot A,\ \cot A\cot B.$$ *(B.U. 1926)*

9. A triangle the lengths of whose sides a, b and c, is placed so that the middle points of its sides are on the co-ordinate axes. Show that the equation to its plane is

$$x/\alpha+y/\beta+z/\gamma=1,$$

where $8\alpha^2=b^2+c^2-a^2$, $8\beta^2=c^2+a^2-b^2$, $8\gamma^2=a^2+b^2-c^2$.

Also show that the co-ordinates of the vertices of the triangle are

$$(-\alpha, \beta, \gamma),\ (\alpha, -\beta, \gamma),\ (\alpha, \beta, -\gamma).$$ *(A.U. 1938)*

10. Show that there are two lines which intersect the lines

$$x-5=\tfrac{1}{5}(y-8)=\tfrac{1}{8}(z-14)$$
$$\tfrac{1}{2}x=\tfrac{1}{4}(y+1)=\tfrac{1}{3}(z-10)$$

and also intersect the x-axis perpendicularly. Find the points in which they meet the x-axis. [*Ans.* (2, 0, 0), (74/17, 0, 0).

11. Taking axis OZ to be vertical, find equations of the line of greatest slope through the point $P(2, -1, 0)$ on the plane

$$2x+3y-4z-1=0.$$ [*Ans.* $\tfrac{1}{8}(x-2)=\tfrac{1}{12}(y+1)=\tfrac{1}{18}z$.

[The required line is the line through P drawn perpendicular to the line of intersection of the given plane and the horizontal plane $z=0$.]

12. The plane $3x+4y+5z=0$ is horizontal. Show that the equations of the line of greatest slope on the plane $x+2y+3z=4$ through the point (2, −5, 4) are

$$(x-2)=(y+5)=-\tfrac{1}{2}(z-4).$$

13. Find the equation of the plane through (0, 1, 1) and (2, 0, −1) which is parallel to the line joining (−1, 1, −2), (3, −2, 4). Find also the perpendicular distance between the line and the plane.

[*Ans.* $6x+10y+z-11=0$; $9/\sqrt{137}$.

14. A straight line is drawn through (α, β, γ) perpendicular to each of two given straight lines which pass through (α, β, γ) and whose direction cosines are l_1, m_1, n_1 ; l_2, m_2, n_2. Show that the volume of the tetrahedron formed by (α, β, γ) and the points where the three lines cut $x=0$ is

$$\alpha^3\sin^2\theta/6l_1l_2(m_1n_2-m_2n_1)$$

where θ is the angle between the lines. *(B.U.)*

15. If OA, OB, OC have direction cosines l_r, m_r, n_r ; $(r=1, 2, 3)$ and OA', OB', OC', bisect the angles BOC, COA, AOB ; the planes AOA', BOB', COC' pass through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3}.$$

16. A point P moves so that three mutually perpendicular lines PA, PB, PC may be drawn cutting the axes OX, OY, OZ at A, B, C and the volume of the tetrahedron $OABC$ is constant and equal to $a^3/6$. Prove that P lies on the surface

$$(x^2+y^2+z^2)^3=8a^3xyz.$$

17. Find the angle between the common line of the planes

$$x+y-z=1,\ 2x-3y+z=2$$

and the line joining the points $(3, -1, 2)$, $(4, 0, -1)$. Find also the equations of the line through the origin which is perpendicular to both the above lines.

[*Ans.* $\cos^{-1}(10/\sqrt{418})$, $x/14=-y/11=z$.

18. Show that the image of the line $x-1=-9(y-2)=-3(z+3)$ in the plane $3x-3y+10z=26$ is the line

$$\tfrac{1}{9}(x-4)=-(y+1)=-\tfrac{1}{3}(z-7).$$

19. The plane $x/a+y/b+z/c=1$ meets the axes at A, B, C respectively and planes are drawn through OX, OY and OZ meeting BC, CA and AB respectively at right angles. Show that these planes are coaxial.

If the common axis meets the plane ABC in P and perpendiculars are drawn from P to the co-ordinate planes, show that the equation of the plane through the feet of the perpendiculars is

$$\frac{x}{bc}+\frac{y}{ca}+\frac{z}{ab}=\frac{2abc}{b^2c^2+c^2a^2+a^2b^2}.$$

20. Prove that

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$$

represents a pair of planes whose line of intersection is equally inclined to the axes. (*C.U. 1927*)

21. From a point P whose co-ordinates are (x, y, z), a perpendicular PM is drawn to the straight line through the origin whose direction cosines are l, m, n, and is produced to P' such the $PM=P'M$.

If the co-ordinates of P' are (x', y', z'), show that

$$\frac{x+x'}{l}=\frac{y+y'}{m}=\frac{z+z'}{n}=2(lx+my+nz).$$ (*P.U.*)

22. Show that the reflection of the plane $2x+3y+z=1$ in the line $x=y/2=z/3$ is the plane $3x-y-26z+7=0$.

23. Prove that the reflection of the plane

$$a'x+b'y+c'z+d'=0$$

in the plane

$$ax+by+cz+d=0$$

is the plane

$$2(aa'+bb'+cc')(ax+by+cz+d)=(a^2+b^2+c^2)(a'x+b'y+c'z+d').$$ (*M.T.*)

24. Find the equations of the straight line through the point (3, 1, 2) to intersect the straight line

$$x+4=(y+1)=2(z-2)$$

and parallel to the plane $4x+y+5z=0$. (*B.U. 1959*)

[*Ans.* $-\tfrac{1}{3}(x-3)=\tfrac{1}{2}(y-1)=-\tfrac{1}{2}(z-2)$.

25. The line $\frac{1}{5}(x+6)=\frac{1}{3}(y+10)=\frac{1}{6}(z+14)$ is the hypotenuse of an isosceles right-angled triangle whose opposite vertex is (7, 2, 4). Find the eqations of the remaining sides. [*Ans.* $\frac{1}{3}(x-7)=\frac{1}{6}(y-2)=\frac{1}{2}(z-4)$; $\frac{1}{2}(x-7)=-\frac{1}{3}(y-2)=\frac{1}{6}(z-4)$.

26. A straight line AB is drawn through a point (4, 1, 7) and perpendicular to the plane $2x+3y-4z=8$. Find the points in which AB and the axis OX are intersected by their common normal. (*B.U. 1926*) [*Ans.* (6, 4, 3), (6, 0, 0).

27. Find the equations of the two straight lines through the origin, each of which intersect the straight line

$$\tfrac{1}{2}(x-3)=(y-3)=z$$

and is inclined at an angle of 60° to it. (*L.U. 1937*)

[*Ans.* $x\tfrac{1}{2}=y=-z$; $x=-y=\tfrac{1}{2}z$.

28. Find the direction cosines of the projection of the line

$$\tfrac{1}{2}(x-1)=-y=(z+2)$$

upon the plane $2x+y-3z=4$. [*Ans.* $2/\sqrt{6}$, $-1/\sqrt{6}$, $1/\sqrt{6}$.

29. Find the equations of the straight line which is the projection on the plane $3x+2y+z=0$, of the line of intersection of the planes

$$3x-y+2z=1,\ x+2y-z=2.$$

[*Ans.* $-(x+1)/11=(y-1)/9=(z-1)/15$.

30. QP, RP are two lines through a point P with direction cosines proportional to 1, 1, -2 and 1, -1, 1 respectively. Find the equation of the plane through the origin which is perpendicular to the plane PQR and parallel to the line QP.

If P is the point $(-1, 1, 1)$, find the co-ordinates of the foot of the perpendicular from P on this plane. [*Ans.* $4x-2y+z=0$, $(-\frac{1}{21}, \frac{11}{21}, \frac{26}{21})$.

31. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $x+y=0$, $y+z=0$, $z+x=0$, $x+y+z=a$ is $2a/\sqrt{6}$ and that the three lines of shortest distance meet at the point $x=y=z=-a$.

32. Prove that the co-ordinates of the points where the shortest distance line between the lines

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \text{ and } \frac{x-a'}{l'}=\frac{y-b'}{m'}=\frac{z-c'}{n'}$$

meets the first line are

$a+l\ \mathrm{cosec}^2\,\theta(u'\cos\theta-u)$, $b+m\ \mathrm{cosec}^2\,\theta(u'\cos\theta-u)$, $c+n\ \mathrm{cosec}^2\,\theta(u'\cos\theta-u)$,

where θ is the angle between the given straight lines

$$u=l(a-a')+m(b-b')+n(c-c'),$$

and

$$u'=l'(a-a')+m'(b-b')+n'(c-c'). \quad (B.U.\ 1920)$$

33. Prove that the shortest distance between the axis of z and the line

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$$

for varying, λ, generates the surface

$$abz(x^2+y^2)=(a^2-b^2)cxy. \quad (B.U.\ 1929)$$

34. Prove that through the point (X, Y, Z) one line can be drawn which intersects the lines $y=x\tan\alpha$, $z=c$; $y=-x\tan\alpha$, $z=-c$ and that it meets the plane XY at the point

$$x=(cYZ\cot\alpha-c^2X)/(Z^2-c^2),\ y=(cXZ\tan\alpha-c^2Y)/(Z^2-c^2),\ z=0. \quad (L.U.)$$

35. Show that the surface generated by a straight line which intersects the lines $y=0$, $z=c$; $x=0$, $z=-c$ and the hyperbola $z=0$, $xy+c^2=0$ is the surface $z^2-xy=c^2$.

36. A straight line intersects the three lines

$$x=0,\ \beta y+\gamma z=\beta\gamma,$$
$$y=0,\ \gamma z+\alpha x=\gamma\alpha,$$
$$z=0,\ \alpha x+\beta y=\alpha\beta.$$

Prove that it is parallel to the plane $x+y+z=0$ and its locus is the surface

$$\Sigma\alpha x^2+\Sigma(\alpha+\beta)zy-\Sigma\alpha(\beta+\gamma)x+\alpha\beta\gamma=0. \quad (M.U.\ 1912)$$

37. Show that the planes

$$x=y\sin\psi+z\sin\varphi,\ y=z\sin\theta+x\sin\psi,\ z=x\sin\varphi+y\sin\theta,$$

intersect in the line

$$\frac{x}{\cos\theta}=\frac{y}{\cos\varphi}=\frac{z}{\cos\psi},$$

if

$$\theta+\varphi+\psi=\tfrac{1}{2}\pi.$$

38. Points P and Q are taken on two given skew lines so that PQ is always parallel to a given plane.

If R divides PQ in a given ratio, prove that the locus of R is a straight line.

39. Find the locus of a point whose distance from a fixed point is in a constant ratio to its distance measured parallel to a given plane, from a given line.

[*Hint.* Take the given plane as XY plane and its intersection with the given line as origin.]

40. Show that the planes

$$2x+3y+4z=6,\ 3x+4y+5z=2,\ x+2y+3z=2$$

form a prism and find the area of its normal section. [*Ans.* $8\sqrt{6}/3$.

41. A straight line meets the co-ordinate planes YOZ, ZOX, XOY in the points A, B, C respectively. If α, β, γ denote the angles BOC, COA, AOB respectively, and if the equation of the plane joining the line to O is $lx+my+nz=0$, show that

$$l^4\cot^2\alpha=m^4\cot^2\beta=n^4\cot^2\gamma. \qquad (M.T.)$$

42. G is the centroid of the triangle whose vertices are the points in which the co-ordinate axes meet a plane α. The perpendicular from G to this plane meets the co-ordinate planes in A, B, C. Prove that

$$\frac{1}{GA}+\frac{1}{GB}+\frac{1}{GC}=\frac{3}{OK},$$

where K is the foot of the perpendicular from the origin O to the plane α.

43. Assuming that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

represents two planes, show that their line of intersection is

$$Fx=Gy=Hz,$$

where F, G, H are the minors of f, g, h in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}.$$

44. Three straight lines mutually at right angles meet in a point P and two of them intersect the axes of x and y respectively, while the third passes through a fixed point $(0, 0, c)$ on the axis of z. Show that the equation of the locus of P is

$$x^2+y^2+z^2=2cz. \qquad (D.U.\ Hons.\ 1944)$$

45. The triangle whose vertices have the rectangular co-ordinates

$$(5, -4, 3),\ (4, -1, -2) \text{ and } (10, -5, 2)$$

respectively is projected orthogonally on to the plane whose equation is $x-y=3$. Find the co-ordinates of the vertices and the area of the new triangle.

(*M.T. 1950*) [*Ans.* $(2, -1, 3)$, $(3, 0, -2)$, $(4, 1, 2)$, $9/\sqrt{2}$.

46. Prove that the plane through the point (α, β, γ) and the line,

$$x=py+q=rz+s$$

is given by

$$\begin{vmatrix} x, & py+q, & rz+s \\ \alpha, & p\beta+q, & r\gamma+s \\ 1, & 1, & 1 \end{vmatrix}=0.$$

(*D. U. Hons., 1955*)

CHAPTER VI

THE SPHERE

6·11. Def. *A* **sphere** *is the locus of a point which remains at a constant distance from a fixed point.*

The constant distance is called the *radius* and the fixed point the *centre* of the sphere.

6·12. Equation of a sphere. Let (a, b, c) be the centre and r the radius of a given sphere.

Equating the radius r to the distance of any point (x, y, z) on the sphere from its centre (a, b, c), we have

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

or $$x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2-r^2)=0 \quad \ldots(A)$$

which is the required equation of the given sphere.

We note the following *characteristics* of the equation (A) of the sphere :

1. It is of the second degree in x, y, z ;
2. The co-efficients of x^2, y^2, z^2 are all equal ;
3. The product terms xy, yz, zx are absent.

Conversely, we shall now show that the general equation

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0,\ a\neq 0 \quad \ldots(B)$$

having the above three characteristics represents a sphere.

The equation (B) can be re-written as

$$\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2},$$

and this manner of re-writing shows that the distance between the variable point (x, y, z) and the fixed point

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$$

is

$$\frac{\sqrt{(u^2+v^2+w^2-ad)}}{a}$$

and is, therefore, constant.

The locus of the equation (B) is thus a sphere.

The radius and, therefore, the sphere is imaginary when

$$u^2+v^2+w^2-ad<0$$

and in this case we call it a *virtual sphere*.

6·13. General equation of a sphere.

The equation (B), when written in the form,

$$x^2+y^2+z^2+\frac{2u}{a}x+\frac{2v}{a}y+\frac{2w}{a}z+\frac{d}{a}=0,\ a\neq 0$$

or $$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0,$$
is taken as the *general equation of a sphere.*

Ex. 1. Find the centres and radii of the spheres :

(*i*) $x^2+y^2+z^2-6x+8y-10z+1 \quad =0.$

(*ii*) $x^2+y^2+z^2+2x-4y-6z+5 \quad =0.$

(*iii*) $2x^2+2y^2+2z^2-2x+4y+2z+3 \quad =0.$

[*Ans.* (*i*) (3, −4, 5) ; 7. (*ii*) (−1, 2, 3) ; 3. (*iii*) $(\frac{1}{2}, -1, -\frac{1}{2})$; 0.

2. Obtain the equation of the sphere described on the join of
$$A(2, -3, 4),\ B(-5, 6, -7)$$
as diameter. [*Ans.* $x^2+y^2+z^2+3(x-y+z)-56=0.$

3. *A point moves so that the sum of the squares of its distances from the six faces of a cube is constant ; show that its locus is a sphere.*

Take the centre of the cube as the origin and the planes through the centre parallel to its faces as co-ordinate planes.

Let each edge of the cube to be equal to $2a$.

Then the equations of the faces of the cube are
$$x=a\ ;\ x=-a\ ;\ y=a\ ,\ y=-a\ ;\ z=a,\ z=-a.$$

If (f, g, h) be any point of the locus, we have
$$(f-a)^2+(f+a)^2+(g-a)^2+(g+a)^2+(h-a)^2+(h+a)^2=k^2 \quad (k, \text{ a constant})$$
or $$2(f^2+g^2+h^2+3a^2)=k^2$$
so that the locus is
$$2(x^2+y^2+z^2+3a^2)=k^2,$$
which is a sphere.

4. A plane passes through a fixed point (a, b, c). Show that the locus of the foot of the perpendicular to it from the origin is the sphere
$$x^2+y^2+z^2-ax-by-cz=0.$$

5. Through a point P three mutually perpendicular straight lines are drawn ; one passes through a fixed point C on the z-axis, while the others intersect the x-axis and y-axis, respectively ; show that the locus of P is a sphere of which C is the centre.

6·2. The sphere through four given points. General equation of a sphere contains *four* effective constants and, therefore, a sphere can be uniquely determined so as to satisfy four conditions, each of which is such that it gives rise to one relation between the constants.

In particular, we can find a sphere through four non-coplanar points
$$(x_1, y_1, z_1),\ (x_2, y_2, z_2),\ (x_3, y_3, z_3),\ (x_4, y_4, z_4).$$

Let
$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \qquad \ldots(i)$$
be the equation of the sphere through the four given points.

We have then the equation
$$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0, \qquad \ldots(ii)$$
and three more similar equations corresponding to the remaining three points.

Eliminating u, v, w, d, from the equation (*i*) and from the four

equations (*ii*) just obtained, we have

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1 \\ x_1^2+y_1^2+z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2, & x_4, & y_4, & z_4, & 1 \end{vmatrix} = 0,$$

which is the equation of the sphere through the four given points.

Note. In numerical questions, we may first find the values of u, v, w, d from the four conditions (*ii*) and then substitute them in the equation (*i*).

Exercises

1. Find the equation of the sphere through the four points

$$(4, -1, 2), (0, -2, 3), (1, -5, -1), (2, 0, 1).$$

[*Ans.* $x^2+y^2+z^2-4x+6y-2z+5=0$.

2. Find the equation of the sphere through the four points

$$(0, 0, 0), (-a, b, c), (a, -b, c), (a, b, -c)$$

and determine its radius. (*D.U. Hons. 1947*)

[*Ans.* $\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}-\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{z}{c}=0$; $\frac{1}{2}(a^2+b^2+c^2)\sqrt{(a^{-2}+b^{-2}+c^{-2})}$.

3. Obtain the equation of the sphere circumscribing the tetrahedron whose faces are

$$x=0, y=0, z=0, x/a+y/b+z/c=1.$$

[*Ans.* $x^2+y^2+z^2-ax-by-cz=0$.

4. Obtain the equation of the sphere which passes through the points

$$(1, 0, 0), (0, 1, 0), (0, 0, 1),$$

and has its radius as small as possible.

[*Ans.* $3(x^2+y^2+z^2)-2(x+y+z)-1=0$.

5. Show that the equation of the sphere passing through the three points $(3, 0, 2), (-1, 1, 1), (2, -5, 4)$ and having its centre on the plane $2x+3y+4z=6$ is $x^2+y^2+z^2+4y-6z=1$.

6. Obtain the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the two points $(0, -2, -4), (2, -1, -1)$.

[*Ans.* $x^2+y^2+z^2-6x-4y+10z+12=0$.

7. A sphere whose centre lies in the positive octant passes through the origin and cuts the planes $x=0, y=0, z=0$, in circles of radii $\sqrt{2}a, \sqrt{2}b, \sqrt{2}c$, respectively ; show that its equation is

$$x^2+y^2+z^2-2\sqrt{(b^2+c^2-a^2)}\,x-2\sqrt{(c^2+a^2-b^2)}\,y-2\sqrt{(a^2+b^2-c^2)}z=0.$$

8. *A plane passes through a fixed point* (a, b, c) *and cuts the axes in* A, B, C. *Show that the locus of the centre of the sphere* $OABC$ *is*

$$a/x+b/y+c/z=2. \qquad (D.U.\ Hons., 1958, 60)$$

Let the sphere $OABC$ be

$$x^2+y^2+z^2+2ux+2vy+2wz=0, \qquad \ldots(1)$$

so that u, v, w are different for different spheres. The points A, B, C where it cuts the three axes are $(-2u, 0, 0), (0, -2v, 0), (0, 0, -2w)$. The equation of the plane ABC is

$$\frac{x}{-2u}+\frac{y}{-2v}+\frac{z}{-2w}=1.$$

Since it passes through (a, b, c) we have

$$\frac{a}{-2u}+\frac{b}{-2v}+\frac{c}{-2w}=1. \qquad \ldots(2)$$

If x, y, z be the centre of the sphere (1),

$$x=-u,\ y=-v,\ z=-w. \qquad \ldots(3)$$

From (2) and (3), we obtain

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$$

as the required locus.

9. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC. (*P.U.* 1940 ; *B.U.* 1955)

[*Ans.* $(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})=4r^2$.

10. If O be the centre of a sphere of radius unity and A, B be two points in a line with O such that

$$OA.OB=1$$

and if P be any variable point on the sphere, show that

$$PA : PB=\text{constant}. \qquad (P.U.\ 1941)$$

11. A sphere of constant radius $2k$ passes through the origin and meets the axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is the sphere

$$x^2+y^2+z^2=k^2.$$

6·31. Plane section of a sphere. *A plane section of a sphere, i.e., the locus of points common to a sphere and a plane, is a circle.*

Let O be the centre of the sphere and P, *any* point on the plane section. Let ON be perpendicular to the given plane ; N being the foot of the perpendicular.

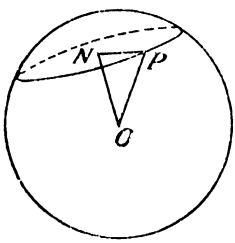

Fig. 23

As ON is perpendicular to the plane which contains the line NP, we have

$$ON \perp NP$$

Hence

$$NP^2=OP^2-ON^2.$$

Now, O and N being fixed points, this relation shows that NP is constant for all positions of P on the section.

Hence the locus of P is a circle whose centre is N, the foot of the perpendicular from the centre of the sphere to the plane.

The section of a sphere by a plane through its centre is known as a *great circle.*

The centre and radius of a great circle are the same as those of the sphere.

Cor. *The circle through three given points lies entirely on any sphere through the same three points.*

Thus the condition of a sphere containing a given circle is equivalent to that of its passing through any three of its points.

6·32. Intersection of two spheres. *The curve of intersection of two spheres is a circle.*

The co-ordinates of points common to any two spheres

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

satisfy both these equations and, therefore, they also satisfy the equation

$$S_1-S_2 \equiv 2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+(d_1-d_2)=0$$

which, being of the first degree, represents a plane.

Thus the points of intersection of the two spheres are the same as those of any one of them and this plane and, therefore, they lie on a circle. [See § 6·31].

6·33. Sphere with a given diameter. *To find the equation of the sphere described on the line joining the points*

$$A(x_1, y_1, z_1),\ B(x_2, y_2, z_2)$$

as diameter.

Let P be any point (x, y, z) on the sphere described on AB as diameter.

Since the section of the sphere by the plane through the three points P, A, B is a great circle having AB as diameter, P lies on a semi-circle and, therefore,

$$PA \perp PB.$$

The direction cosines of PA, PB are proportional to

$$x-x_1,\ y-y_1,\ z-z_1 \text{ and } x-x_2,\ y-y_2,\ z-z_2$$

respectively. Therefore they will be perpendicular, if

$$(\mathbf{x}-\mathbf{x}_1)(\mathbf{x}-\mathbf{x}_2)+(\mathbf{y}-\mathbf{y}_1)(\mathbf{y}-\mathbf{y}_2)+(\mathbf{z}-\mathbf{z}_1)(\mathbf{z}-\mathbf{z}_2)=0$$

which is the required equation of the sphere.

Ex. Show that the condition for the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

to cut the sphere

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d=0$$

in a great circle is

$$2uu_1+2vv_1+2ww_1-(d+d_1)=2r_1^2.$$

where r_1 is the radius of the latter sphere.

6·4. Equations of a circle. Any circle is the intersection of its plane with some sphere through it. Therefore a circle can be represented by *two* equations, one being of a sphere and the other of the plane.

Thus the two equations

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,\ lx+my+nz=p$$

taken together represent a circle.

A circle can also be represented by the equations of any two spheres through it.

Note.—The student may note that the equations

$$x^2+y^2+2gx+2fy+c=0,\ z=0$$

also represent a circle which is the intersection of the cylinder

$$x^2+y^2+2gx+2fy+c=0,$$

with the plane

$$z=0.$$

Examples

1. *Find the equations of the circle circumscribing the triangle formed by the three points*

$$(a, 0, 0),\ (0, b, 0),\ (0, 0, c).$$

Obtain also the co-ordinates of the centre of this circle.

The equation of the plane passing through these three points is

$$x/a + y/b + z/c = 1.$$

The required circle is the curve of intersection of this plane with *any* sphere through the three points.

To find the equation of this sphere, a fourth point is necessary, which, for the sake of convenience, we take as origin.

If

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

be the sphere through these four points, we have

$$a^2+2ua+d=0;\ b^2+2vb+d=0;\ c^2+2wc+d=0;$$
$$d=0.$$

These give

$$d=0,\ u=-\tfrac{1}{2}a,\ v=-\tfrac{1}{2}b,\ w=-\tfrac{1}{2}c.$$

Thus the equation of the sphere is

$$x^2+y^2+z^2-ax-by-cz=0.$$

Hence the equations of the circle are

$$x^2+y^2+z^2-ax-by-cz=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

To find the centre of this circle, we obtain the foot of the perpendicular from the centre $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ of the sphere to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

The equations of the perpendicular are

$$\frac{x-\frac{1}{2}a}{1/a}=\frac{y-\frac{1}{2}b}{1/b}=\frac{z-\frac{1}{2}c}{1/c}=r, \text{ say}$$

so that

$$\left(\frac{r}{a}+\frac{a}{2},\ \frac{r}{b}+\frac{b}{2},\ \frac{r}{c}+\frac{c}{2}\right),$$

is any point on the line. Its intersection with the plane is given by

$$r\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{2}=0 \text{ or } r=-\frac{1}{(2\Sigma a^{-2})}.$$

Thus the centre is

$$\left[\frac{a(b^{-2}+c^{-2})}{2\Sigma a^{-2}},\ \frac{b(c^{-2}+a^{-2})}{2\Sigma a^{-2}},\ \frac{c(a^{-2}+b^{-2})}{2\Sigma a^{-2}}\right].$$

2. *Show that the centre of all sections of the sphere*

$$x^2+y^2+z^2=r^2$$

by planes through a point (x', y', z') *lie on the sphere*

$$x(x-x')+y(y-y')+z(z-z')=0.$$

The plane which cuts the sphere in a circle with centre (f, g, h) is

$$f(x-f)+g(y-g)+h(z-h)=0.$$

It will pass through (x', y', z'), if

$$f(x'-f)+g(y'-g)+h(z'-h)=0,$$

and accordingly the locus of (f, g, h) is the sphere

$$x(x'-x)+y(y'-y)+z(z'-z)=0.$$

Exercises

1. Find the centre and the radius of the circle

$$x+2y+2z=15,\ x^2+y^2+z^2-2y-4z=11.$$

[*Ans.* (1, 3, 4), $\sqrt{7}$.

2. Find the equation of that section of the sphere

$$x^2+y^2+z^2=a^2,$$

of which a given internal point (x_1, y_1, z_1) is the centre. (*P.U. 1939 Suppl.*)

(The plane through (x_1, y_1, z_1) drawn perpendicular to the line joining this point to the centre (0, 0, 0) of the sphere determines the required section.)

3. Obtain the equations of the circle lying on the sphere

$$x^2+y^2+z^2-2x+4y-6z+3=0$$

and having its centre at (2, 3, −4).

[*Ans.* $x^2+y^2+z^2-2x+4y-6z+3=0=x+5y-7z-45$.

4. O is the origin and A, B, C, are the points

$$(4a, 4b, 4c),\ (4b, 4c, 4a),\ (4c, 4a, 4b).$$

Show that the sphere

$$x^2+y^2+z^2-2(x+y+z)(a+b+c)+8(bc+ca+ab)=0$$

passes through the nine-point circles of the faces of the tetrahedron $OABC$.

5. Find the equation of the diameter of the sphere $x^2+y^2+z^2=29$ such that a rotation about it will transfer the point (4, −3, 2) to the point (5, 0, −5) along a great circle of the sphere. Find also the angle through which the sphere must be so rotated. (*L.U.*) [*Ans.* $\frac{1}{2}x=\frac{1}{5}y=\frac{1}{5}z$, $\cos^{-1}(16/29)$.

6. Show that the following points are concyclic :—

(*i*) (5, 0, 2), (2, −6, 0), (7, −3, 8), (4, −9, 6).

(*ii*) (−8, 5, 2), (−5, 2, 2), (−7, 6, 6), (−4, 3, 6).

6·41. Spheres through a given circle. The equation

$$S+\lambda U=0,$$

obviously represents a general sphere passing through the circle with equations

$$S=0,\ U=0,$$

where

$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d,$$
$$U\equiv lx+my+nz-p.$$

Also, the equation

$$S+\lambda S'=0$$

represents a general sphere through the circle with equations

$$S=0,\ S'=0,$$

where

$$S\equiv x^2+y^2+z^2+2ux+2vy+2wz+d,$$
$$S'\equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'.$$

Here λ is an arbitrary constant which may be so chosen that these equations fulfil one more condition.

Note 1. We notice that the equation of the plane of the circle through the two given spheres is

$$S-S' \equiv 2(u-u')x+2(v-v')y+2(w-w')z+d-d'=0.$$

From this we see that the equation of any sphere through the circle

$$S=0,\ S'=0$$

can also be taken of the form

$$S+k(S-S')=0;$$

k, being any arbitrary constant.

This form sometimes proves comparatively more convenient.

Note 2. It is important to remember that the general equation of a sphere through the circle

$$x^2+y^2+2gx+2fy+c=0,\ z=0$$

is

$$x^2+y^2+z^2+2gx+2fy+kz+c=0,$$

where k is different for the different spheres.

Examples

1. *Find the equation of the sphere through the circle*

$$x^2+y^2+z^2=9,\ \ 2x+3y+4z=5$$

and the point (1, 2, 3).

The sphere

$$x^2+y^2+z^2-9+\lambda(2x+3y+4z-5)=0$$

passes through the given circle for all values of λ.

It will pass through (1, 2, 3) if

$$5+15\lambda=0 \text{ or } \lambda=-\tfrac{1}{3}.$$

The required equation of the sphere, therefore, is

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0.$$

2. *Show that the two circles*

$$x^2+y^2+z^2-y+2z=0,\ x-y+z-2=0;$$

$$x^2+y^2+z^2+x-3y+z-5=0,\ 2x-y+4z-1=0;$$

lie on the same sphere and find its equation. (*D.U. Hons. 1947*)

The equation of *any* sphere through the first circle is

$$x^2+y^2+z^2-y+2z+\lambda(x-y+z-2)=0, \qquad \ldots(i)$$

and that of *any* sphere through the second circle is

$$x^2+y^2+z^2+x-3y+z-5+\mu(2x-y+4z-1)=0. \qquad \ldots(ii)$$

The equations (i) and (ii) will represent the same sphere, if λ, μ can be chosen so as to satisfy the four equations

$$\lambda=2\mu+1,$$
$$-1-\lambda=-\mu-3,$$
$$2+\lambda=4\mu+1,$$
$$-2\lambda=-\mu-5.$$

The first two of these equations give $\lambda=3$, $\mu=1$, and these values clearly satisfy the remaining two equations also. These four equations

in λ, μ being consistent, the two circles lie on the same sphere, *viz.*,

$$x^2+y^2+z^2-y+2z+3(x-y+z-2)=0,$$

i.e.,

$$x^2+y^2+z^2+3x-4y+5z-6=0.$$

Exercises

1. Find the equation of the sphere through the circle

$$x^2+y^2+z^2+2x+3y+6=0,\ x-2y+4z-9=0$$

and the centre of the sphere

$$x^2+y^2+z^2-2x+4y-6z+5=0.$$

[*Ans.* $x^2+y^2+z^2+7y-8z+24=0$.

2. Show that the equation of the sphere having its centre on the plane

$$4x-5y-z=3$$

and passing through the circle with equations

$$x^2+y^2+z^2-2x-3y+4z+8=0,\ x^2+y^2+z^2+4x+5y-6z+2=0;$$

is

$$x^2+y^2+z^2+7x+9y-11z-1=0.$$

3. Obtain the equation of the sphere having the circle

$$x^2+y^2+z^2+10y-4z-8=0,\ x+y+z=3$$

as the great circle.

[The centre of the required sphere lies on the plane $x+y+z=3$.]

[*Ans.* $x^2+y^2+z^2-4x+6y-8z+4=0$.

4. A sphere S has points (0, 1, 0), (3, −5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of S with the plane

$$5x-2y+4z+7=0$$

as a great circle.

[*Ans.* $x^2+y^2+z^2+2x+2y+2z+2=0$.

5. Obtain the equation of the sphere which passes through the circle $x^2+y^2=4$, $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

[*Ans.* $x^2+y^2+z^2\pm6z-4=0$.

6. Show that the two circles

$$2(x^2+y^2+z^2)+8x-13y+17z-17=0,\ 2x+y-3z+1=0;$$
$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0;$$

lie on the same sphere and find its equation.

[*Ans.* $x^2+y^2+z^2+5x-6y+7z-8=0$.

7. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0;$$

lie on the sphere and find its equation. (*D.U. Hons., 1945*)

[*Ans.* $x^2+y^2+z^2-2x-2y-2z-6=0$.

6·5. Intersection of a sphere and a line.

Let

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \ldots(1)$$

and

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \ldots(2)$$

be the equations of a sphere and a line respectively.

The point $(lr+\alpha, mr+\beta, nr+\gamma)$ which lies on the given line (2) for all values of r, will also lie on the given sphere (1), if r satisfies the equation

$$r^2(l^2+m^2+n^2)+2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0, \quad \ldots(A)$$

and this latter being a quadratic equation in r, gives two values say, r_1, r_2 of r. Then

$$(lr_1+\alpha, mr_1+\beta, nr_1+\gamma), \ (lr_2+\alpha, mr_2+\beta, nr_2+\gamma)$$

are the two points of intersection.

Thus every straight line meets a sphere in two points which may be real, imaginary or coincident.

Ex. Find the co-ordinates of the points where the line

$$\tfrac{1}{4}(x+3)=\tfrac{1}{3}(y+4)=-\tfrac{1}{5}(z-8)$$

intersects the sphere

$$x^2+y^2+z^2+2x-10y=23.$$

[*Ans.* $(1, -1, 3)$; $(5, 2, -2)$.

6·51. Power of a point. If l, m, n, are the actual direction cosines of the given line (2) in § 6·5, so that $l^2+m^2+n^2=1$, then r_1, r_2, are the distances of the point A (α, β, γ) from the points of intersection P and Q.

$$\therefore \quad AP \times AQ = r_1 r_2 = \alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d$$

which is independent of the direction cosines, l, m, n.

Thus *if from a fixed point A, chords be drawn in any direction to intersect a given sphere in P and Q, then $AP.AQ$ is constant.* This constant is called the *power* of the point A with respect to the sphere.

Example

Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular straight lines through a fixed point is constant.

Take the fixed point O as the origin and *any* three mutually perpendicular lines through it as the co-ordinate axes. With this choice of axes, let the equation of the given sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

The x-axis, $(y=0=z)$ meets the sphere in points given by

$$x^2+2ux+d=0,$$

so that if x_1, x_2 be its roots, the two points of intersection are

$$(x_1, 0, 0), \ (x_2, 0, 0).$$

Also we have

$$x_1+x_2=-2u, \ x_1x_2=d.$$

$$\therefore \quad (\text{intercept on } x\text{-axis})^2=(x_1-x_2)^2$$
$$=(x_1+x_2)^2-4x_1x_2=4(u^2-d).$$

Similarly

$$(\text{intercept on } y\text{-axis})^2=4(v^2-d),$$
$$(\text{intercept on } z\text{-axis})^2=4(w^2-d).$$

The sum of the squares of the intercepts

$$=4(u^2+v^2+w^2-3d)$$
$$=4(u^2+v^2+w^2-d)-8d$$
$$=4r^2-8p,$$

where r is the radius of the given sphere and p is the power of the given point with respect to the sphere.

Since the sphere and the point are both given, r and p are both constants.

Hence the result.

Note. The co-efficients u, v, w and d in the equation of the sphere will be different for different sets of mutually perpendicular lines through O as axes.

Since, however, the sphere is fixed and the point O is also fixed, the expressions

$$r^2=u^2+v^2+w^2-d$$

for the square of the radius and

$$p=d,$$

for the power of the point, *w.r.t.* the sphere will be invariants.

Exercises

1. Find the locus of a point whose powers with respect to two given spheres are in a constant ratio.

2. Show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chords.

6·6. Equation of tangent plane. *To find the equation of the tangent plane at any point* (α, β, γ) *of the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

As (α, β, γ) lies on the sphere, we have

$$\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0. \qquad \ldots(i)$$

The points of intersection of any line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \qquad \ldots(ii)$$

through (α, β, γ) with the given sphere are

$$(lr+\alpha, mr+\beta, nr+\gamma)$$

where the values of r are the roots of the equation

$$r^2(l^2+m^2+n^2)+2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0.$$

By virtue of the condition (i), one root of this quadratic equation is zero so that one of the points of intersection coincides with (α, β, γ).

In order that the second point of intersection may also coincide with (α, β, γ), the second value of r must also vanish and this requires

$$l(\alpha+u)+m(\beta+v)+n(\gamma+w)=0. \qquad \ldots(iii)$$

Thus the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

meets the sphere in two coincident points at (α, β, γ) and so is a *tangent line* to it thereat for any set of values of l, m, n which satisfy the condition (*iii*).

The locus of the tangent lines at (α, β, γ) is, thus, obtained by eliminating l, m, n between (*iii*) and the equations (*ii*) of the line and this gives

$$(x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)=0$$

or
$$\alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d$$
$$=\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0, \text{ from } (i)$$

which is a plane known as the *tangent plane* at (α, β, γ).

Hence

$$(\alpha+u)x+(\beta+v)y+(\gamma+w)z+(u\alpha+v\beta+w\gamma+d)=0$$

is the equation of the tangent plane to the given sphere at (α, β, γ).

Cor. 1. *The line joining the centre of a sphere to any point on it is perpendicular to the tangent plane thereat,* for the direction cosines of the line joining

the centre $(-u, -v, -w)$ to the point (α, β, γ)

on the sphere are proportional to

$$(\alpha+u, \beta+v, \gamma+w)$$

which are also the co-efficients of x, y, z in the equation of the tangent plane at (α, β, γ).

Cor. 2. If a plane or a line touch a sphere, then the length of the perpendicular from its centre to the plane or the line is equal to its radius.

Note. Any line in the tagent plane through its point of contact touches the section of the sphere by any plane through that line.

Examples

1. *Show that the plane* $lx+my+nz=p$ *will touch the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

if

$$(ul+vm+wn+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d).$$

Equating the radius $\sqrt{(u^2+v^2+w^2-d)}$ of the sphere to the length of the perpendicular from the centre $(-u, -v, -w)$ to the plane

$$lx+my+nz=p,$$

we get the required condition.

2. *Find the two tangent planes to the sphere*

$$x^2+y^2+z^2-4x+2y-6z+5=0$$

which are parallel to the plane

$$2x+2y=z.$$

The general equation of a plane parallel to the given plane

$$2x+2y-z=0,$$

is

$$2x+2y-z+\lambda=0.$$

This will be a tangent plane. if its distance from the centre (2, −1, 3) of the sphere is equal to the radius 3 and this requires

$$\frac{-1+\lambda}{\pm 3}=3.$$

Thus

$$\lambda=10 \text{ or } -8.$$

Hence the required tangent planes are

$$2x+2y-z+10=0, \text{ and } 2x+2y-z-8=0.$$

3. *Find the equation of the sphere which touches the sphere*

$$x^2+y^2+z^2-x+3y+2z-3=0,$$

at (1, 1, −1) *and passes through the origin.*

The tangent plane to the given sphere at (1, 1, −1) is

$$x+5y-6=0.$$

The equation of the required sphere is, therefore, of the form

$$x^2+y^2+z^2-x+3y+2z-3+k(x+5y-6)=0.$$

This will pass through the origin if

$$k=-\tfrac{1}{2}.$$

Thus the required equation is

$$2(x^2+y^2+z^2)-3x+y+4z=0.$$

4. *Find the equations of the sphere through the circle*

$$x^2+y^2+z^2=1, \quad 2x+4y+5z=6$$

and touching the plane

$$z=0.$$

The sphere

$$x^2+y^2+z^2-1+\lambda(2x+4y+5z-6)=0$$

passes through the given circle for all values of λ.

Its centre is $(-\lambda, -2\lambda, -\frac{5}{2}\lambda)$, and radius is

$$\sqrt{(\lambda^2+4\lambda^2+\tfrac{25}{4}\lambda^2+1+6\lambda)}.$$

Since it touches $z=0$, we have by Cor. 2,

$$-\tfrac{5}{2}\lambda=\pm\sqrt{(5\lambda^2+\tfrac{25}{4}\lambda^2+1+6\lambda)}.$$

or

$$5\lambda^2+6\lambda+1=0.$$

This gives

$$\lambda=-1 \text{ or } -\tfrac{1}{5}.$$

The two corresponding spheres, therefore, are

$$x^2+y^2+z^2-2x-4y-5z+5=0,$$

$$5(x^2+y^2+z^2)-2x-4y-5z+1=0.$$

5. *Find the equations of the two tangent planes to the sphere*

$$x^2+y^2+z^2=9,$$

which passes through the line

$$x+y=6, \quad x-2z=3.$$

Any plane

$$x+y-6+\lambda(x-2z-3)=0$$

through the given line will touch the given sphere if

$$\pm\frac{-6-3\lambda}{\sqrt{[(1+\lambda)^2+1+4\lambda^2]}}=3,$$

or

$$2\lambda^2-\lambda-1=0.$$

This gives

$$\lambda=1,\ -\tfrac{1}{2}.$$

The two corresponding planes, therefore, are

$$2x+y-2z=9,\ x+2y+2z=9.$$

Exercises

1. Find the equation of the tangent plane to the sphere

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

at the point (1, 2, 3).

[*Ans.* $4x+9y+14z-64=0$.

2. Find the equations of the tangent line to the circle

$$x^2+y^2+z^2+5x-7y+2z-8=0,\ 3x-2y+4z+3=0$$

at the point (−3, 5, 4).

[*Ans.* $(x+3)/32=(y-5)/34=-(z-4)/7$.

3. Find the value of a for which the plane

$$x+y+z=a\sqrt{3}$$

touches the sphere

$$x^2+y^2+z^2-2x-2y-2z-6=0.$$

[*Ans.* $\sqrt{3}\pm 3$.

4. Show that the plane $2x-2y+z+12=0$ touches the sphere

$$x^2+y^2+z^2-2x-4y+2z=3$$

and find the point of contact.

[*Ans.* (−1, 4, −2).

[The point of contact of a tangent plane is the point where the line through the centre perpendicular to the plane meets the sphere.]

5. Find the co-ordinates of the points on the sphere

$$x^2+y^2+z^2-4x+2y=4$$

the tangent planes at which are parallel to the plane

$$2x-y+2z=1.$$

[*Ans.* (4, −2, 2), (0, 0, −2).

6. Show that the equation of the sphere which touches the sphere

$$4(x^2+y^2+z^2)+10x-25y-2z=0,$$

at (1, 2, −2) and passes through the point (−1, 0, 0) is

$$x^2+y^2+z^2+2x-6y+1=0.$$

7. Obtain the equations of the tangent planes to the sphere

$$x^2+y^2+z^2+6x-2z+1=0$$

which pass through the line

$$3(16-x)=3z=2y+30.$$

[*Ans.* $2x+2y-z-2=0,\ x+2y-2z+14=0$.

8. Obtain the equations of the sphere which pass through the circle

$$x^2+y^2+z^2-2x+2y+4z-3=0,\ 2x+y+z=4$$

and touch the plane $2x+4y=14$.

[*Ans.* $x^2+y^2+z^2+2x+4y+6z-11=0,\ x^2+y^2+z^2-2x+2y-4z-3=0$.

9. Find the equation of the sphere which has its centre at the origin and which touches the line

$$2(x+1)=2-y=z+3.$$

[*Ans.* $9(x^2+y^2+z^2)=5$.

10. Find the equation of a sphere of radius r which touches the three co-ordinate axes. How many spheres can be so drawn.

[*Ans.* $2(x^2+y^2+z^2)+2\sqrt{2}(\pm x\pm y\pm z)r+r^2=0$; eight.

11. Prove that the equation of the sphere which lies in the octant $OXYZ$ and touches the co-ordinate planes is of the form

$$x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0.$$

Show that, in general, two spheres can be drawn through a given point to touch the co-ordinate planes and find for what positions of the point the spheres are

(*a*) real, (*b*) coincident. (*P.U. 1944*)

[The distances of the centre from the co-ordinate planes are all equal to the radius so that we may suppose that λ is the radius and $(\lambda, \lambda, \lambda)$ is the centre ; λ being the parameter.]

12. Show that the spheres

$$x^2+y^2+z^2=25$$

$$x^2+y^2+z^2-24x-40y-18z+225=0$$

touch externally and find the point of the contact.

[*Ans.* (12/5, 20/5, 9/5).

13. Find the centres of the two spheres which touch the plane

$$4x+3y=47$$

at the point (8, 5, 4) and which touch the sphere

$$x^2+y^2+z^2=1.$$

[*Ans.* (4, 2, 4). (64/21, 27/21, 4).

14. Obtain the equations of spheres that pass through the points (4, 1, 0), (2, −3, 4), (1, 0, 0) and touch the plane $2x+2y-z=11$. (*P.U. 1934*)

[*Ans.* $x^2+y^2+z^2-6x+2y-4z+5=0$; $16(x^2+y^2+z^2)-102x+50y-49z+86=0$.

15. Find the equation of the sphere inscribed in the tetrahedron whose faces are

(*i*) $x=0$, $y=0$, $z=0$, $x+2y+2z=1$.

(*ii*) $x=0$, $y=0$, $z=0$, $2x-6y+3z+6=0$.

[*Ans.* (*i*) $32(x^2+y^2+z^2)-8(x+y+z)+1=0$. (*ii*) $9(x^2+y^2+z^2)+6(x-y+z)+2=0$.

16. Tangent plane at any point of the sphere $x^2+y^2+z^2=r^2$ meets the co-ordinate axes at A, B, C. Show that the locus of the point of intersection of planes drawn parallel to the co-ordinate planes through A, B, C is the surface $x^{-2}+y^{-2}+z^{-2}=r^{-2}$.

6·61. Plane of Contact. *To find the locus of the points of contact of the tangent planes which pass through a given point* (α, β, γ).

Let (x', y', z') be any point on the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

The tangent plane

$$x(x'+u)+y(y'+v)+z(z'+w)+(ux'+vy'+wz'+d)=0,$$

at this point will pass through (α, β, γ),

if

$$\alpha(x'+u)+\beta(y'+v)+\gamma(z'+w)+(ux'+vy'+wz'+d)=0,$$

or

$$x'(\alpha+u)+y'(\beta+v)+z'(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0,$$

which is the condition that the point (x', y', z') should lie on the plane

$$x(\alpha+u)+y(\beta+v)+z(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0.$$

It is called the *plane of contact* for the point (α, β, γ).

Thus the locus of points of contact is the circle in which the plane cuts the sphere.

Ex. 1. Show that the line joining any point P to the centre of a given sphere is perpendicular to the plane of contact of P and if OP meets it in Q, then

$$OP.OQ=(\text{radius})^2.$$

Ex. 2. Show that the planes of contact of all points on the line

$$x/2=(y-a)/3=(z+3a)/4$$

with respect to the sphere $x^2+y^2+z^2=a^2$ pass through the line

$$-(2x+3a)/13=(y-a)/3=z/1.$$

6 62. The polar plane. *If a variable line is drawn through a fixed point A meeting a given sphere in P, Q and point R is taken on this line such that the points A, R divide this line internally and externally in the same ratio, then the locus of R is a plane called polar plane of A w.r. to the sphere.*

It may be easily seen that if the points A, R divide PQ internally and externally in the same ratio, then the points P, Q also divide AR internally and externally in the same ratio.

Consider the sphere

$$x^2+y^2+z^2=a^2, \qquad \ldots(1)$$

and let A be the point (α, β, γ).

Let (x, y, z) be the co-ordinates of the point R on any line through A. The co-ordinates of the point dividing AR in the ratio $\lambda : 1$ are

$$\left[\left(\frac{\lambda x+\alpha}{\lambda+1}\right), \left(\frac{\lambda y+\beta}{\lambda+1}\right), \left(\frac{\lambda z+\gamma}{\lambda+1}\right)\right].$$

This point will be on the sphere (1) for values of λ which are roots of the quadratic equation

$$\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2+\left(\frac{\lambda y+\beta}{\lambda+1}\right)^2+\left(\frac{\lambda z+\gamma}{\lambda+1}\right)^2=a^2,$$

i.e.,

$$\lambda^2(x^2+y^2+z^2-a^2)+2\lambda(\alpha x+\beta y+\gamma z-a^2)+(\alpha^2+\beta^2+\gamma^2-a^2)=0 \qquad \ldots(2)$$

Its roots λ_1 and λ_2 are the ratios in which the points P, Q divide AR.

Since P, Q divide AR internally and externally in the same ratio, we have

$$\lambda_1+\lambda_2=0.$$

Thus from (2), we have

$$\alpha x+\beta y+\gamma z-a^2=0, \qquad \ldots(3)$$

which is the relation satisfied by the co-ordinates (x, y, z) of R.

Hence (3) is the locus of R. Clearly it is a plane.

Thus we have seen here that the equation of the polar plane of

the point (α, β, γ) with respect to the sphere

$$x^2+y^2+z^2=a^2,$$

is

$$\alpha x+\beta y+\gamma z=a^2.$$

It may similarly be shown that the polar plane of (α, β, γ) with respect to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

is

$$(\alpha+u)x+(\beta+v)y+(\gamma+w)z+(u\alpha+v\beta+w\gamma+d)=0.$$

On comparing the equation of the polar plane with that of the tangent plane (§ 6·6) and the plane of contact (§ 6·61), we see that the polar plane of a point lying on the sphere is the tangent plane at the point and that of a point, lying outside it, is its plane of contact.

If π be the polar plane of a point P, then P is called the pole of the plane π.

6·63. Pole of a plane. *To find the pole of the plane*

$$lx+my+nz=p \quad \ldots (i)$$

with respect to the sphere

$$x^2+y^2+z^2=a^2.$$

If (α, β, γ) be the required pole, then we see that the equation (i) is identical with

$$\alpha x+\beta y+\gamma z=a^2 \quad \ldots (ii)$$

so that, on comparing (i) and (ii), we obtain

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{\gamma}{n}=\frac{a^2}{p},$$

or

$$\alpha=a^2l/p,\ \beta=a^2m/p,\ \gamma=a^2n/p.$$

Thus

$$(a^2l/p,\ a^2m/p,\ a^2n/p)$$

is the required pole.

6·64. Some results concerning poles and polars. In the following discussion, we always take the equation of a sphere in the form

$$x^2+y^2+z^2=a^2.$$

1. *The line joining the centre O of a sphere to any point P is perpendicular to the polar plane of P.*

The direction ratios of the line joining the centre O (0, 0, 0) to the point $P(\alpha, \beta, \gamma)$ are α, β, γ and these are also the direction ratios of the normal to the polar plane $\alpha x+\beta y+\gamma z=a^2$ of $P(\alpha, \beta, \gamma)$.

2. *If the line joining the centre O of a sphere to any point P meets the polar plane of P in Q, then*

$$OP.OQ=a^2,$$

where a is the radius of the sphere.

We have,

$$OP=\sqrt{(\alpha^2+\beta^2+\gamma^2)},$$

Also, OQ, which is the length of the perpendicular from the centre $O(0, 0, 0)$ to the polar plane $\alpha x+\beta y+\gamma z=a^2$ of P, is given by

$$OQ=\frac{a^2}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}}.$$

Hence the result.

3. *If the polar plane of a point P passes through another point Q, then the polar plane of Q passes through P.*

The condition that the polar plane

$$\alpha_1 x+\beta_1 y+\gamma_1 z=a^2,$$

of $P(\alpha_1, \beta_1, \gamma_1)$ passes through $Q(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=a^2,$$

which is also, by symmetry, or directly, the condition that the polar plane of Q passes through P.

Conjugate points. *Two points such that the polar plane of either passes through the other are called conjugate points.*

4. *If the pole of a plane π_1 lies on another plane π_2, then the pole of π_2 also lies on π_1.*

The condition that the pole

$$\left(\frac{a^2l_1}{p_1}, \frac{a^2m_1}{p_1}, \frac{a^2n_1}{p_1}\right)$$

of the plane π_1

$$l_1x+m_1y+n_1z=p_1$$

lies on the plane π_2

$$l_2x+m_2y+n_2z=p_2$$

is

$$a^2(l_1l_2+m_1m_2+n_1n_2)=p_1p_2$$

which is also, clearly, the condition that the pole

$$(a^2l_2/p_2, a^2m_2/p_2, a^2n_2/p_2)$$

of π_2 lies on π_1.

Conjugate planes. *Two planes such that the pole of either lies on the other are called conjugate planes.*

5. *The polar planes of all the points on a line l pass through another line l'.*

The polar plane of any point,

$$(lr+\alpha, mr+\beta, nr+\gamma),$$

on the line, l,

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is

$$(lr+\alpha)x+(mr+\beta)y+(nr+\gamma)z=a^2,$$

or

$$(\alpha x+\beta y+\gamma z-a^2)+r(lx+my+nz)=0,$$

which clearly passes through the line

$$\alpha x+\beta y+\gamma z-a^2=0,\ lx+my+nz=0,$$

whatever value, r, may have. Hence the result.

Let this line be l'. We shall now prove that the polar plane of every point on l' passes through l.

Now, as the polar plane of any arbitrary point P on l passes through *every* point of l', therefore, the polar plane of every point of l' passes through the point of P on l and as, P is arbitrary, it passes through every point of l *i.e.*, it passes through l.

Thus we see that if l' is the line such that the polar planes, of all the points on a line l, pass through it, then the polar planes of all the points on l' pass through l.

Polar Lines. *Two lines such that the polar plane of every point on either passes through the other are called* **Polar Lines.**

Exercises

1. Show that polar line of

$$(x+1)/2=(y-2)/3=(z+3),$$

with respect to the sphere

$$x^2+y^2+z^2=1,$$

is the line

$$\frac{7x+3}{11}=\frac{2-7y}{5}=\frac{z}{-1}.$$

2. Show that if a line l is coplanar with the polar line of a line l', then, l' is coplanar with the polar line of l.

3. If PA, QB be drawn perpendicular to the polars of Q and P respectively, with respect to a sphere, centre O, then

$$\frac{PA}{QB}=\frac{OP}{OQ}.$$

4. Show that, for a given sphere, there exist an unlimited number of tetrahedra such that each vertex is the pole of the opposite face with respect to the sphere.

(Such a tetrahedron is known as a *self-conjugate* or *self-polar* tetrahedron)

6·7. Angle of Intersection of two spheres.

Def. *The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point* and is, therefore, also equal to the angle between the radii of the spheres to the common point ; the radii being perpendicular to the respective tangent planes at the point.

The angle of intersection at every common point of the spheres is the same, for if P, P' be any two common points and C, C' the centres of the spheres, the triangles $CC'P$ and $CC'P'$ are congruent and accordingly

$$\angle CPC'=\angle CP'C'.$$

The spheres are said to be **orthogonal** if the angle of intersection of two spheres is a right angle. In this case

$$CC'^2=CP^2+C'P^2.$$

6·71. Condition for the orthogonality of two spheres.

To find the condition for the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

to be orthogonal.

The spheres will be orthogonal if *the square of the distance between their centres is equal to the sum of the squares of their radii* and this requires

$$(u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_2)^2 = (u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2)$$

or

$$\mathbf{2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2.}$$

Exercises

1. Find the equation of the sphere that passes through the circle

$$x^2+y^2+z^2-2x+3y-4z+6=0,\ 3x-4y+5z-15=0$$

and cuts the sphere

$$x^2+y^2+z^2+2x+4y-6z+11=0$$

orthogonally.

[*Ans.* $5(x^2+y^2+z^2)-13x+19y-25z+45=0$.

2. Find the equation of the sphere that passes through the two points

$$(0, 3, 0),\ (-2, -1, -4)$$

and cuts orthogonally the two spheres

$$x^2+y^2+z^2+x-3z-2=0,\ 2(x^2+y^2+z^2)+x+3y+4=0.$$

[*Ans.* $x^2+y^2+z^2+2x-2y+4z-3=0$.

3. Find the equation of the sphere which touches the plane

$$3x+2y-z+2=0$$

at the point (1, −2, 1) and cuts orthogonally the sphere

$$x^2+y^2+z^2-4x+6y+4=0.$$ (*L.U.*)

[*Ans.* $x^2+y^2+z^2+7x+10y-5z+12=0$.

4. Show that every sphere through the circle

$$x^2+y^2-2ax+r^2=0,\ z=0,$$

cuts orthogonally every sphere through the circle

$$x^2+z^2=r^2,\ y=0.$$

5. Two points P, Q are conjugate with respect to a sphere S; show that the sphere on PQ as diameter cuts S orthogonally.

6. If two spheres S_1 and S_2 are orthogonal, the polar plane of any point on S_1 with respect to S_2 passes through the other end of the diameter of S_1 through P.

Example

Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is

$$r_1r_2/\sqrt{(r_1^2+r_2^2)}.$$

Let the common circle be

$$x^2+y^2=a^2,\ z=0.$$

The general equation of the sphere through this circle being

$$x^2+y^2+z^2+2kz-a^2=0,$$

let the two given spheres through the circle be

$$x^2+y^2+z^2+2k_1z-a^2=0,\ x^2+y^2+z^2+2k_2z-a^2=0.$$

We have

$$r_1^2=k_1^2+a^2,\ r_2^2=k_2^2+a^2.$$ ...(*i*)

Since the spheres cut orthogonally, we have

$$2k_1k_2=a^2+a^2=2a^2.$$ ...(*ii*)

From (*i*) and (*ii*), eliminating k_1, k_2, we have

$$(r_1^2-a^2)(r_2^2-a^2)=a^4,$$

or

$$a^2=r_1^2r_2^2/(r_1^2+r_2^2).$$

Hence the result.

6·8. Radical plane. *To show that the locus of points whose powers with respect to two spheres are equal is a plane perpendicular to the line joining their centres.*

The powers of the point $P(x, y, z)$ with respect to the spheres

$$S_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$S_2=x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$

are

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1,$$

and

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2,$$

respectively.

Equating these, we obtain

$$\mathbf{2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+(d_1-d_2)=0,}$$

which is the required locus, and being of the first degree in (x, y, z), it represents a plane which is obviously perpendicular to the line joining the centres of the two spheres and is called the *radical plane* of the two spheres.

Thus *the radical plane of the two spheres*

$$S_1=0,\ S_2=0,$$

in both of which the co-efficients of the second degree terms are equal to unity, is

$$S_1-S_2=0.$$

In case the two spheres intersect, the plane of their common circle is their radical plane. (§ 6·32).

6·81. Radical line. *The three radical planes of three spheres taken two by two intersect in a line.*

If

$$S_1=0,\ S_2=0,\ S_3=0$$

be the three spheres, their radical planes

$$S_1-S_2=0,\ S_2-S_3=0,\ S_3-S_1=0,$$

clearly meet in the line

$$S_1=S_2=S_3.$$

This line is called the *radical line* of the three spheres.

6·82. Radical Centre. *The four radical lines of four spheres taken three by three intersect at a point.*

The point common to the three planes

$$S_1=S_2=S_3=S_4$$

is clearly common to the radical lines, taken three by three, of the four spheres

$$S_1=0,\ S_2=0,\ S_3=0,\ S_4=0.$$

This point is called the *radical centre* of the four spheres.

6·83. Theorem. *If* $S_1=0$, $S_2=0$, *be two spheres, then the equation*

$$S_1+\lambda S_2=0,$$

λ *being the parameter, represents a system of spheres such that any two members of the system have the same radical plane.*

Let

$$S_1+\lambda_1 S_2=0 \text{ and } S_1+\lambda_2 S_2=0,$$

be any two members of the system.

Making the co-efficients of second degree terms unity, we write them in the form

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}=0,\ \frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0.$$

The radical plane of these two spheres is

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0,$$

or

$$S_1-S_2=0.$$

Since this equation is independent of λ_1 and λ_2, we see that every two members of the system have the same radical plane.

Co-axal System. Def. *A system of spheres such that any two members thereof have the same radical plane is called a co-axal system of spheres.*

Thus the system of spheres

$$S_1+\lambda S_2=0$$

is co-axal and we say that it is determined by the two spheres

$$S_1=0,\ S_2=0.$$

The common radical plane is

$$S_1-S_2=0$$

This co-axal system is also given by the equation

$$S_1+k_2(S_1-S_2)=0.$$

Refer Note 1, § 6·41, *P.* 92.

Note. It can similarly be proved that the system of spheres

$$S+\lambda U=0$$

is co-axal ; $S=0$ being a sphere and $U=0$ a plane ; the common radical plane is $U=0$.

Cor. *The locus of the centres of spheres of a co-axal system is a line.*

For, if (x, y, z) be the centre of the sphere

$$S_1+\lambda S_2=0,$$

we have

$$x=-\frac{u_1+\lambda u_2}{1+\lambda},\quad y=-\frac{v_1+\lambda v_2}{1+\lambda},\quad z=-\frac{w_1+\lambda w_2}{1+\lambda}.$$

On eliminating λ, we find that it lies on the line

$$\frac{x+u_1}{u_1-u_2}=\frac{y+v_1}{v_1-v_2}=\frac{z+w_1}{w_1-w_2}.$$

This result is also otherwise clear as the line joining the centres of any two spheres is perpendicular to their common radical plane.

6·9. A simplified form of the equation of the two spheres.

By taking the line joining the centres of two given spheres as X-axis, their equations take the form

$$x^2+y^2+z^2+2u_1x+d_1=0,\ x^2+y^2+z^2+2u_2x+d_2=0.$$

Their radical plane is

$$2x(u_1-u_2)+(d_1-d_2)=0.$$

Further, if we take the radical plane as the YZ plane, *i.e.*, $x=0$, we have $d_1=d_2=d$, (say).

Thus by taking the line joining the centres as X-axis and the radical plane as the YZ plane, the equations of *any* two spheres can be put in the simplified form

$$x^2+y^2+z^2+2u_1x+d=0,\ x^2+y^2+z^2+2u_2x+d=0,$$

where u_1, u_2 are different.

Cor. 1. The equation

$$x^2+y^2+z^2+2kx+d=0$$

represents a co-axal system of spheres for different values of k; d being constant. The YZ plane is the common radical plane and X-axis is the line of centres.

Cor. 2. Limiting points. The equation

$$x^2+y^2+z^2+2kx+d=0$$

can be written as

$$(x+k)^2+y^2+z^2=k^2-d.$$

For $k=\pm\sqrt{d}$, we get spheres of the system with radius zero and thus the *system includes the two point spheres*

$$(-\sqrt{d},\ 0,\ 0),\ (\sqrt{d},\ 0,\ 0).$$

These two points are called the *limiting points* and are real only when d is positive, *i.e.*, when the spheres do not meet the radical plane in a real circle.

Def. Limiting points *of a co-axal system of spheres are the point spheres of the system.*

Examples

1. *Find the limiting points of the co-axal system defined by the spheres*

$$x^2+y^2+z^2+3x-3y+6=0,\ x^2+y^2+z^2-6y-6z+6=0.$$

The equation of the plane of the circle through the two given spheres is

$$3x+3y+6z=0, \text{ i.e., } x+y+2z=0.$$

Then the equation of the co-axal system determined by given spheres is

$$x^2+y^2+z^2+3x-3y+6+\lambda(x+y+2z)=0,$$

i.e., $$x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0. \quad \ldots(1)$$

The centre of (1) is

$$\left[-\frac{3+\lambda}{2}, -\frac{\lambda-3}{2}, -\lambda\right],$$

and radius is

$$\sqrt{\left[\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6\right]}.$$

Equating this radius to zero, we obtain

$$6\lambda^2-6=0,$$

i.e., $$\lambda=\pm 1.$$

The spheres corresponding to these values of λ become point spheres coinciding with their centres and are the limiting points of the given system of spheres.

The limiting points, therefore, are

$$(-1, 2, 1) \text{ and } (-2, 1, -1).$$

2. *Show that spheres which cut two given spheres along great circles all pass through two fixed points.* (*P.U. 1944 Suppl.*)

With proper choice of axes, the equations of the given spheres take the form

$$x^2+y^2+z^2+2u_1x+d=0, \quad \ldots(i)$$

$$x^2+y^2+z^2+2u_2x+d=0. \quad \ldots(ii)$$

The equation of any other sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad \ldots(iii)$$

where u, v, w, c are different for different spheres.

The plane

$$2x(u-u_1)+2vy+2wz+(c-d)=0,$$

of the circle common to (i) and (iii) will pass through the centre

$$(-u_1, 0, 0)$$

of (i), if

$$-2u_1(u-u_1)+(c-d)=0, \quad \ldots(iv)$$

which is thus the condition for the sphere (iii) to cut the sphere (i) along a great circle.

Similarly

$$-2u_2(u-u_2)+(c-d)=0, \quad \ldots(v)$$

is the condition for the sphere (iii) to cut the sphere (ii) along a great circle.

Solving (iv) and (v) for u and c, we get

$$u=u_1+u_2\,;\; c=2u_1u_2+d,$$

so that u, c are constants, being dependent on u_1, u_2, d only.

The sphere (*iii*) cuts X-axis at points whose x-co-ordinates are the roots of the equation

$$x^2+2ux+c=0.$$

The roots of this equation are constant, depending as they do upon the constants u and c only.

Thus every sphere (*iii*) meets the X-axis at the same two points and hence the result.

Exercises

1. Show that the sphere

$$x^2+y^2+z^2+2vy+2wz-d=0$$

passes through the limiting points of the co-axal system

$$x^2+y^2+z^2+2kx+d=0$$

and cuts every member of the system orthogonally, whatever be the values of v, w.

Hence deduce that every sphere that passes through the limiting points of a co-axal system cuts every sphere of that system orthogonally.

2. Show that the locus of the point spheres of the system

$$x^2+y^2+z^2+2vy+2wz-d=0$$

is the common circle of the system

$$x^2+y^2+z^2+2ux+d=0\ ;$$

u, v, w being the parameters and d a constant.

3. Show that the sphere which cuts two spheres orthogonally will cut every member of the co-axal system determined by them orthogonally.

4. Find the limiting points of the co-axal system of spheres

$$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0.$$

[*Ans.* $(2, -3, 4)$; $(-2, 3, -4)$.

5. Three spheres of radii r_1, r_2, r_3, have their centres A, B, C at the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $r_1^2+r_2^2+r_3^2=a^2+b^2+c^2$. A fourth sphere passes through the origin and A, B, C. Show that the radical centre of the four spheres lies on the plane $ax+by+cz=0$. (*D.U.*)

6. Show that the locus of a point from which equal tangents may be drawn to the three spheres

$$x^2+y^2+z^2+2x+2y+2z+2=0,$$
$$x^2+y^2+z^2+4x+4z+4=0,$$
$$x^2+y^2+z^2+x+6y-4z-2=0,$$

is the straight line

$$x/2=(y-1)/5=z/3.$$

7. Show that there are, in general, two spheres of a co-axal system which touch a given plane.

Find the equations to the two spheres of the co-axal system

$$x^2+y^2+z^2-5+\lambda(2x+y+3z-3)=0,$$

which touch the plane

$$3x+4y=15.$$

[*Ans.* $x^2+y^2+z^2+4x+2y+6z-11=0$, $5(x^2+y^2+z^2)-8x-4y-12z-13=0$.

8. P is a variable point on a given line and A, B, C are its projections on the axes. Show that the sphere $OABC$ passes through a fixed circle.

9. Show that the radical planes of the sphere of a co-axal system and of any given sphere pass through a line.

CHAPTER VII

THE CONE AND CYLINDER

7·1. Def. *A* **cone** *is a surface generated by a straight line which passes through a fixed point and satisfies one more condition* : for instance, it may intersect a given curve or touch a given surface.

The fixed point is called the *vertex* and the given curve the *guiding curve* of the cone.

Any individual straight line on the surface of a cone is called its *generator*.

In this book we shall be concerned only with **quadric cones,** *i.e.*, cones with second degree equations.

7·12. Equation of a cone with a conic as guiding curve. *To find the equation of the cone whose vertex is at the point*

$$(\alpha,\ \beta,\ \gamma)$$

and whose generators intersect the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0. \qquad \ldots (i)$$

The equations to *any* line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots (ii)$$

This line meets the plane $z=0$ in the point

$$\left(\alpha-\frac{l\gamma}{n},\ \beta-\frac{m\gamma}{n},\ 0\right)$$

which will lie on the given conic, if

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+$$

$$2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0. \qquad \ldots (iii)$$

This is the condition for the line (ii) to intersect the conic (i). Eliminating l, m, n between (ii) and (iii), we get

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+$$

$$b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+c=0$$

or $a(\alpha z-x\gamma)^2+2h(\alpha z-x\gamma)(\beta z-y\gamma)+(\beta z-y\gamma)^2+$

$$2g(\alpha z-x\gamma)(z-\gamma)+2f(\beta z-y\gamma)(z-\gamma)+c(z-\gamma)^2=0,$$

which is the required equation of the cone.

Note. The degree of the equation of a cone depends upon the nature of the guiding curve. In case the guiding curve is a conic, the equation of the cone shall be of the second degree, as is seen above. Cones with second degree

equations are called *Quadric cones.* In what follows, we shall almost be exclusively concerned with quadric cones only.

Exercises

1. Find the equation of the cone whose generators pass through the point (α, β, γ) and have their direction cosines satisfying the relation

$$al^2+bm^2+cn^2=0. \qquad (P.U.\ 1937)$$

$$[Ans.\ \ a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0.$$

2. Find the equation of the cone whose vertex is the point (1, 1, 0) and whose guiding curve is

$$y=0,\ x^2+z^2=4.$$

$$[Ans.\ \ x^2-3y^2+z^2-2xy+8y-4=0.$$

3. Obtain the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic

$$x^2/a^2+y^2/b^2=1,\ z=0.$$

$$\left[Ans.\ \left(\frac{\alpha z-x\gamma}{a}\right)^2+\left(\frac{\beta z-y\gamma}{b}\right)^2=(z-\gamma)^2.\right.$$

4. The section of a cone whose vertex is P and guiding curve the ellipse $x^2/a^2+y^2/b^2=1,\ z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1.$$

5. Show that the equation of the cone whose vertex is the origin and whose base is the circle through the three points

$$(a, 0, 0),\ (0, b, 0),\ (0, 0, c)$$

is $$\Sigma a(b^2+c^2)yz=0. \qquad (B.U.\ 1958)$$

6. Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve is the circle

$$x^2+y^2+z^2=4,\ x+y+z=1.$$

$$[Ans.\ \ 5x^2+3y^2+z^2-2xy-6yz-4zx+6x+8y+10z=26.$$

7. The plane $lx+my+nz=0$ moves in such a way that its intersection with the planes

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0$$

are perpendicular. Show that the normal to the plane at the origin describes, in general, a cone of the second degree, and find its equation. Examine the case in which $aa'+bb'+cc'=0$. $(M.T.\ 1956)$

7·13. Enveloping cone of a sphere. *To find the equation of the cone whose vertex is at the point* (α, β, γ) *and whose generators touch the sphere*

$$x^2+y^2+z^2=a^2. \qquad \ldots(i)$$

The equations to *any* line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(ii)$$

The points of intersection of the line (ii) with the sphere (i) are given by

$$r^2(l^2+m^2+n^2)+2r(l\alpha+m\beta+n\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)=0.$$

(See § 6·5)

and the line will touch the sphere, if the two values of r are coincident, and this requires

$$(l\alpha+m\beta+n\gamma)^2=(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2). \qquad \ldots(iii)$$

This is the condition for the line (ii) to touch the sphere (i).

Eliminating l, m, n, between (ii) and (iii), we get

$$[\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)]^2 = [(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2][\alpha^2+\beta^2+\gamma^2-a^2], \quad \ldots(iv)$$

which is the required equation of the cone.

If we write

$$S \equiv x^2+y^2+z^2-a^2,\ S_1 \equiv \alpha^2+\beta^2+\gamma^2-a^2,\ T \equiv \alpha x+\beta x+\gamma z-a^2,$$

the equation (iv) can be re-written as

$$(T-S_1)^2 = (S-2T+S_1)S_1$$

or

$$SS_1 = T^2,$$

i.e.,
$$(x^2+y^2+z^2-a^2)(\alpha^2+\beta^2+\gamma^2-a^2) = (\alpha x+\beta y+\gamma z-a^2)^2.$$

Def. Enveloping cone. *The cone formed by the tangent lines to a surface, drawn from a given point is called the enveloping cone of the surface with given point as its vertex.*

Exercises

1. Find the enveloping cone of the sphere

$$x^2+y^2+z^2-2x+4z=1$$

with its vertex at (1, 1, 1).

[*Ans.* $4x^2+3y^2-5z^2-6yz-8x+16z-4=0$.

2. Show that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has its vertex at (2, 4, 1) in a rectangular hyperbola.

7·14. Quadric cones with vertex at origin. *To prove that the equation of a cone with its vertex at the origin is homogeneous in x, y, z and conversely.*

We take up the general equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \ldots(1)$$

of the second degree and show that if it represents a cone with its vertex at the origin, then

$$u=v=w=d=0.$$

Let $P(x', y', z')$ be any point on the cone represented by the equation (1).

Now, rx', ry', rz' are the general co-ordinates of a point on the line joining P to the origin O.

Since OP is a generator of the cone (1), the point

$$(rx', ry', rz')$$

should lie on it for every value of r. Hence

$$r^2(ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y')+2r(ux'+vy'+wz')+d=0,$$

must be an identity.

This gives

$$ax'^2+by'^2+cz'^2+2fy'z'+2gz'x'+2hx'y'=0, \quad \ldots(i)$$

$$ux'+vy'+wz'=0, \quad \ldots(ii)$$

$$d=0. \quad \ldots(iii)$$

From (iii),

$$d=0.$$

From (*ii*), we see that if u, v, w, be not all zero, then the co-ordinates x', y', z', of any point on the cone satisfy an equation of the first degree so that the surface is a plane and we have a contradiction.

Hence

$$u=v=w=0.$$

Thus we see that the equation of a cone with its vertex at the origin, is necessarily homogeneous.

Conversely, *every homogeneous equation of the second degree represents a cone with its vertex at the origin.*

It is clear from the nature of the equation that if the co-ordinates x', y', z', satisfy it, then so do also rx', ry', rz', for all values of r.

Hence if any point P lies on the surface, then every point on OP and, therefore, the entire line OP lies on it.

Thus the surface is generated by lines through the origin O and hence, by definition, it is a cone with its vertex at O.

Note. A homogeneous equation of the second degree will represent a pair of planes, if the homogeneous expression can be factorized into linear factors. The condition for this has already been obtained in Chapter II.

Cor. 1. If l, m, n be the direction ratios of any generator of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \qquad \ldots(1)$$

then any point (lr, mr, nr) on the generator lies on it and, therefore,

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0. \qquad \ldots(2)$$

Conversely, it is obvious that if the result (2) be true then the line with direction ratios l, m, n is a generator of the cone whose equation is (1).

Cor. 2. The general equation of the cone with its vertex at (α, β, γ) is

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2+2f(z-\gamma)(y-\beta)+2g(x-\alpha)(z-\gamma)+2h(x-\alpha)(y-\beta)=0,$$

as can easily be verified by transferring the origin to the point (α, β, γ).

Example

Find the equation of the quadric cone whose vertex is at the origin and which passes through the curve given by the equations

$$ax^2+by^2+cz^2=1,\ lx+my+nz=p.$$

The required equation is the homogeneous equation of the second degree satisfied by points satisfying the two given equations. We rewrite

$$lx+my+nz=p$$

as

$$\frac{lx+my+nz}{p}=1.$$

Thus the required equation is

$$ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{p}\right)^2,$$

or

$$\Sigma(ap^2-l^2)x^2-2\Sigma lmxy=0.$$

Exercises

1. Find the equation of the cone with vertex at the origin and direction cosines of its generators satisfying the relation

$$3l^2-4m^2+5n^2=0.$$

[*Ans.* $3x^2-4y^2+5z^2=0$.

2. Find the equations to the cones with vertex at the origin and which pass through the curves given by the equations

(*i*) $z=2,\ x^2+y^2=4$.

(*ii*) $ax^2+by^2=2z,\ lx+my+nz=p$.

(*iii*) $x^2+y^2+z^2+x-2y+3z=4\ ;\ x^2+y^2+z^2+2x-3y+4z=5$.

[*Ans.* (*i*) $x^2+y^2-z^2=0$. (*ii*) $p(ax^2+by^2)=2z(lx+my+nz)$.

(*iii*) $2x^2+y^2-5xy-3yz+4zx=0$.

3. A sphere S and a plane α have, respectively, the equations

$$\varphi+u+c=0\ ;\ v=1,$$

where $\varphi=x^2+y^2+z^2$, u and v are homogeneous linear functions of x, y, z and c is a constant. Find the equation of the cone whose generators join the origin O to the points of intersection of S and α.

Show that this cone meets S again in points lying on a plane β and find the equation of β in terms of u, v and c.

If the radius of S varies, while its centre, the plane α, and the point O remain fixed, prove that β passes through a fixed line. (*M.T.*)

[The required cone, C, is given by

$$C\equiv\varphi+uv+cv^2.$$

Now
$$C-S\equiv(\varphi+uv+cv^2)-(\varphi+u+c)=(v-1)(u+cv+c)$$

so that we see that the cone C meets S again in points lying on the plane $\beta\equiv u+cv+c=0$.

Since the radius of S varies and its centre remains fixed, we see that u is constant while c varies. Also v is constant. This shows that the plane $\beta\equiv u+c(v+1)$ passes through the line of intersection of the fixed planes $u=0$, $v+1=0$.]

7·15. Determination of Cones under given conditions. As the general equation of a quadric cone with a given vertex contains *five arbitrary constants*, it follows that five conditions determine such a cone provided each condition gives rise to a single relation between the constants. For instance, *a cone can be determined so as to have any given five concurrent lines as generators* provided no three of them are co-planar.

Examples

1. *Show that the general equation to a cone which passes through the three axes is*

$$fyz+gzx+hxy=0.$$

The general equation of a cone with its vertex at the origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \qquad \ldots(i)$$

Since X-axis is a generator, its direction cosines 1, 0, 0 satisfy (*i*). This gives $a=0$. Similarly $b=c=0$.

2. *Show that a cone can be found so as to contain any two given sets of three mutually perpendicular concurrent lines as generators.*

Take the three lines of one set as co-ordinate axes.

Let the lines OP, OQ, OR of the second set be

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1},\quad \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2},\quad \frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3},$$

respectively.

The general equation of a cone through the three axes is

$$fyz+gzx+hxy=0.$$

It will contain the lines OP and OQ as generators, if

$$fm_1n_1+gn_1l_1+hl_1m_1=0, \qquad ...(i)$$

$$fm_2n_2+gn_2l_2+hl_2m_2=0. \qquad ...(ii)$$

The lines of the set being mutually perpendicular, we have

$$\left.\begin{aligned} m_1n_1+m_2n_2+m_3n_3&=0,\\ n_1l_1+n_2l_2+n_3l_3&=0,\\ l_1m_1+l_2m_2+l_3m_3&=0. \end{aligned}\right\} \quad .. (A)$$

Adding (i), (ii) and employing the relation (A), we deduce the condition

$$fm_3n_3+gn_3l_3+hl_3m_3=0,$$

so that the cone through OP and OQ passes through OR also.

Exercises

1. Find the equation to the cone which passes through the three co-ordinate axes as well as the lines

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3},\quad \frac{x}{3}=\frac{y}{-1}=\frac{z}{1}.$$

[*Ans.* $3yz+16zx+15xy=0$.

2. Find the equation of the cone which contains the three co-ordinate axes and the lines through the origin with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2).

[*Ans.* $\Sigma l_1l_2(m_1n_2-m_2n_1)yz=0$.

3. Find the equation of the quadric cone which passes through the three co-ordinate axes and the three mutually perpendicular lines

$$\tfrac{1}{2}x=y=-z,\ x=\tfrac{1}{3}y=\tfrac{1}{5}z,\ \tfrac{1}{8}x=-\tfrac{1}{11}y=\tfrac{1}{5}z.$$

[*Ans.* $16yz-33zx-25xy=0$.

7·2. Condition that the general equation of the second degree should represent a cone. Co-ordinates of the vertex.

We have seen that the equation of a cone with its vertex at the origin is necessarily homogeneous and conversely. Thus any given equation of the second degree will represent a cone if, and only if, there is a point such that on transferring the origin to the same the equation becomes homogeneous.

Let

$$f(x, y, z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy \\ +2ux+2vy+2wz+d=0, \qquad ...(1)$$

represent a cone having its vertex at (x', y', z').

Shift the origin to the vertex (x', y', z') so that we change

$$x \text{ to } x+x',\ y \text{ to } y+y' \text{ and } z \text{ to } z+z'.$$

The transformed equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+$$
$$2[x(ax'+hy'+gz'+u)+y(hx'+by'+fz'+v)+z(gx'+fy'+cz'+w)]$$
$$+f(x', y', z')=0. \quad \ldots(2)$$

The equation (2) represents a cone with its vertex at the origin and must, therefore, be homogeneous. This gives

$$ax'+hy'+gz'+u=0, \quad \ldots (i)$$
$$hx'+by'+fz'+v=0, \quad \ldots (ii)$$
$$gx'+fy'+cz'+w=0, \quad \ldots (iii)$$
$$f(x', y', z')=0, \quad \ldots (iv)$$

Also, $f(x', y', z')\equiv x'(ax'+hy'+gz'+u)+y'(hx'+by'+fz'+v)+$
$$z'(gx'+fy'+cz'+w)+(ux'+vy'+wz'+d).$$

Thus with the help of (i), (ii) and (iii), we see that (iv) is equivalent to

$$ux'+vy'+wz'+d=0. \quad \ldots (v)$$

Eliminating x', y', z' between (i), (ii), (iii), and (v), we obtain

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix} = 0,$$

as the required *condition for the general equation* (1) *of the second degree to represent a cone.*

Assuming that the condition is satisfied, the co-ordinates (x', y', z') *of the vertex are obtained by solving simultaneously the three linear equations* (i), (ii) *and* (iii).

Cor. If $F(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+$
$$2ux+2vy+2wz+d=0,$$
represents a cone, the co-ordinates of its vertex satisfy the equations

$$F_x=0,\ F_y=0,\ F_z=0,\ F_t=0,$$

where 't' is used to make $F(x, y, z)$ *homogeneous and is put equal to unity after differentiation.*

Making $F(x, y, z)$ homogeneous, we write

$$F(x, y, z, t)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2uxt+2vyt+2wzt+dt^2.$$

Then

$$F_x=2(ax+hy+gz+ut),\ F_y=2(hx+by+fz+vt),$$
$$F_z=2(gx+fy+cz+wt),\ F_t=2(ux+vy+wz+dt).$$

Putting $t=1$, we see from (i), (ii), (iii), (v) that the vertex (x_1, y_1, z_1) satisfies the equations

$$F_x=0,\ F_y=0,\ F_z=0,\ F_t=0.$$

Note. The student should note that the co-efficients of second degree terms in the transformed equation (2) are the same as those in the original equation (1).

Note. The equation $F(x, y, z)=0$ represents a cone if, and only if, the four linear equations $F_x=0$, $F_y=0$, $F_z=0$, $F_t=0$ are consistent. In the case of consistency the vertex is given by any three of these.

Example

Prove that the equation

$$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0,$$

represents a cone whose vertex is $(-1, -2, -3)$.

Making the equation homogeneous, we obtain

$$F(x, y, z, t)=4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2.$$

Equating to zero the partial derivatives with respect to x, y, z and t, we obtain the four linear equations

$$8x+2y+12t=0, \quad \ldots (i)$$
$$2x-3y-3z+11t=0, \quad \ldots (ii)$$
$$-3y+4z+6t=0, \quad \ldots (iii)$$
$$12x-11y+6z+8t=0. \quad \ldots (iv)$$

Replacing t by unity and solving the resulting three linear equations (i), (ii), (iii) for x, y, z, we obtain

$$x=-1, \; y=-2, \; z=-3.$$

The values satisfy (iv) also.

Thus the equation is a cone with vertex $(-1, -2, -3)$.

Exercises

1. Prove that the equation

$$x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y-2z-20=0,$$

represents a cone with vertex $(1, -2, 3)$.

2. Prove that the equation

$$2y^2-8yz-4zx-8xy+6x-4y-2z+5=0,$$

represents a cone whose vertex is $(-7/6, 1/3, 5/6)$.

Example

Find the equations to the lines in which the plane

$$2x+y-z=0,$$

cuts the cone

$$4x^2-y^2+3z^2=0.$$

Let

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

be the equations of any one of the two lines in which the given plane meets the given cone so that we have

$$2l+m-n=0, \; 4l^2-m^2+3n^2=0.$$

These two equations have now to be solved for l, m, n. Eliminating n, we have

$$4l^2-m^2+3(2l+m)^2=0,$$

i.e.,

$$8l^2+6lm+m^2=0.$$

$$\therefore \quad \frac{l}{m}=\frac{-6\pm\sqrt{36-32}}{16}=-\frac{1}{4} \text{ or } -\frac{1}{2}.$$

Also we have

$$2\frac{l}{m}+1-\frac{n}{m}=0.$$

$$\therefore \quad \text{when } \frac{l}{m}=-\frac{1}{4}, \text{ we have } \frac{n}{m}=\frac{1}{2},$$

so that

$$\frac{l}{-1}=\frac{m}{4}=\frac{n}{2},$$

and when

$$\frac{l}{m}=-\frac{1}{2}, \text{ we have } \frac{n}{m}=0$$

so that

$$\frac{l}{-1}=\frac{m}{2}=\frac{n}{0}.$$

Thus the two required lines are

$$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}, \quad \frac{x}{-1}=\frac{y}{2}=\frac{z}{0}.$$

Exercises

1. Find the equations of the lines of intersection of the following planes and cones :

(*i*) $x+3y-2z=0$, $x^2+9y^2-4z^2=0$.
(*ii*) $3x+4y+z=0$, $15x^2-32y^2-7z^2=0$.
(*iii*) $x+7y-5z=0$, $3yz+14zx-30xy=0$.

[*Ans.* (*i*) $x=2z, y=0$; $3y=2z, x=0$. (*ii*) $\frac{x}{-3}=\frac{y}{2}=\frac{z}{1}$, $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}$

(*iii*) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$, $\frac{x}{3}=\frac{y}{1}=\frac{z}{2}$.

2. Show that the equation of the quadric cone which contains the three co-ordinate axes and the lines in which the plane

$$x-5y+3z=0$$

cuts the cone

$$7x^2+5y^2-3z^2=0,$$

is

$$yz+10zx-18xy=0.$$

3. Find the angle between the lines of intersection of

(*i*) $x-3y+z=0$ and $x^2-5y^2+z^2=0$.
(*ii*) $10x+7y-6z=0$ and $20x^2+7y^2-108z^2=0$.
(*iii*) $4x-y-5z=0$ and $8yz+3zx-5xy=0$.
(*iv*) $x+y+z=0$ and $6xy+3yz-2zx=0$.
(*v*) $x+y+z=0$ and $x^2-yz+xy-3z^2=0$.

[*Ans.* $\cos^{-1}(5/6)$, (*ii*) $\cos^{-1}(16/21)$, (*iii*) $\cos^{-1}(2\sqrt{2}/3)$, (*iv*) $\pi/3$, (*v*) $\pi/6$.

4. Prove that the plane

$$ax+by+cz=0$$

cuts the cone

$$yz+zx+xy=0,$$

in perpendicular lines if

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0.$$

(*D.U. Hons. 1955*)

[Refer, also Ex. 1, after the next section 7·3]

7·3. Mutually perpendicular generators of a cone.

To find the condition that the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad ...(i)$$

should have three mutually perpendicular generators.

Let

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu} \qquad ...(ii)$$

be any generator of the cone so that

$$a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu=0. \qquad ...(iii)$$

Equation of the plane through the origin perpendicular to the line (*ii*) is

$$\lambda x+\mu y+\nu z=0. \qquad ...(iv)$$

If l, m, n be the direction cosines of any one of the generators in which the plane cuts the given cone, we have

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0, \qquad ...(v)$$

and

$$l\lambda+m\mu+n\nu=0. \qquad ...(vi)$$

Eliminating n between (*v*) and (*vi*), we obtain

$$l^2(a\nu^2+c\lambda^2-2g\lambda\nu)+2lm(c\lambda\mu+h\nu^2-g\mu\nu+f\lambda\nu)+m^2(b\nu^2+c\mu^2-2f\mu\nu)=0$$

which, being a quadratic in $l : m$, we see that the plane (*iv*) cuts the given cone in two generators.

Hence if $(l_1, m_1, n_1,)$, (l_2, m_2, n_2) be the direction cosines of these two generators, we have

$$\frac{l_1l_2}{m_1m_2}=\frac{b\nu^2+c\mu^2-2f\mu\nu}{a\nu^2+c\lambda^2-2g\lambda\nu},$$

or

$$\frac{l_1l_2}{b\nu^2+c\mu^2-2f\mu\nu}=\frac{m_1m_2}{a\nu^2+c\lambda^2-2g\lambda\nu}.$$

From symmetry, each of these is further

$$=\frac{n_1n_2}{a\mu^2+b\lambda^2-2h\lambda\mu}=k, \text{ (say)}$$

$$\therefore \quad l_1l_2+m_1m_2+n_1n_2=k[a(\mu^2+\nu^2)+b(\nu^2+\lambda^2)+c(\lambda^2+\mu^2)-2f\mu\nu-2g\nu\lambda-2h\lambda\mu]$$

$$=k(a+b+c)(\lambda^2+\mu^2+\nu^2), \qquad ...(vii)$$

with the help of (*iii*).

If these two generators be at rt. angles, we have

$$l_1l_2+m_1m_2+n_1n_2=0,$$

and hence

$$\mathbf{a}+\mathbf{b}+\mathbf{c}=0.$$

Also conversely, if $a+b+c=0$, we see from (*vii*) that these two generators are at right angles.

Since $x/\lambda=y/\mu=z/\nu$, is any arbitrary generator and the condition obtained is independent of λ, μ, ν, we see that *if*

$$a+b+c=0,$$

then the plane through the vertex perpendicular to any generator of the cone cuts it in two other perpendicular generators. These two generators will themselves be perpendicular to the first generator so that the three generators will be perpendicular in pairs.

Thus if

$$a+b+c=0$$

the cone has an infinite number of sets of three mutually perpendicular generators.

In fact if this condition is satisfied, then the plane perpendicular to *any* generator OP of the cone cuts the same in two perpendicular generators OQ, OR, so that OP, OQ, OR is a set of three mutually perpendicular generators.

Note. If the general equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone having sets of three mutually perpendicular generators, then also

$$a+b+c=0$$

for, on shifting the origin to its vertex, the co-efficients of the second degree terms remain unaffected.

Exercises

1. Show that the two straight lines represented by the equations

$$ax+by+cz=0 \text{ and } yz+zx+xy=0,$$

will be perpendicular if

$$1/a+1/b+1/c=0. \qquad (P.U.\ 1939)$$

[The sum of the co-efficients of x^2, y^2 and z^2 in the equation of the given cone being zero, we see that the given plane will cut the given cone in perpendicular generators if the normal to the plane through the vertex which is the origin, *viz.*,

$$x/a=y/b=z/c$$

is a generator of the cone.

This requires

$$bc+ca+ab=0, \textit{ i.e.,}\ 1/a+1/b+1/c=0.]$$

2. Prove that the plane $lx+my+nz=0$ cuts the cone

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$$

in perpendicular lines if

$$(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0.$$

3. If

$$x=\tfrac{1}{2}y=z$$

represents one of a set of three mutually perpendicular generators of the cone

$$11yz+6zx-14xy=0,$$

find the equations of the other two.

$$[Ans.\quad \frac{x}{2}=\frac{y}{-3}=\frac{z}{4}\ ;\ \frac{x}{-11}=\frac{y}{2}=\frac{z}{7}.$$

4. If

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

represent one of a set of three mutually perpendicular generators of the cone

$$5yz-8zx-3xy=0,$$

find the equations of the other two. *(D.U. Hons. 1960)*

$$[Ans.\ \ \frac{x}{5}=\frac{y}{-4}=\frac{z}{1}\ ;\ \frac{x}{1}=\frac{y}{1}=\frac{z}{-1}.$$

5. Show that the cone whose vertex is the origin and which passes through the curve of intersection of the surface $2x^2-y^2+2z^2=3d^2$ and any plane at a distance d, from the origin has three mutually perpendicular generators.

6. Find the locus of a point from which three mutually perpendicular lines can be drawn to intersect the central conic

$$ax^2+by^2=1,\ z=0.$$

$$[Ans.\ \ a(x^2+z^2)+b(y^2+z^2)=1.$$

7. Show that three mutually perpendicular tangent lines can be drawn to the sphere

$$x^2+y^2+z^2=r^2$$

from any point on the sphere

$$2(x^2+y^2+z^2)=3r^2.$$

8. Three points P, Q, R are taken on the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere. *(P.U. 1949)*

7·4. Intersection of a line with a cone.

To find the points of intersection of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

with the cone

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \qquad \ldots(ii)$$

The point $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$ which lies on the line (i) for all values of r will lie on the cone (ii) for values of r given by the equation

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2+2f(mr+\beta)(nr+\gamma)$$
$$+2g(lr+\alpha)(nr+\gamma)+2h(lr+\alpha)(mr+\beta)=0,$$

or
$$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$$
$$+2r[l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)]$$
$$+f(\alpha,\ \beta,\ \gamma)=0, \qquad \ldots(A)$$

which being a *quadratic* in r, we see that there are *two* points of intersection.

Hence *every line meets a quadric cone in two points.*

Cor. A plane section of a quadric cone is a conic, as every line in the plane meets the curve of intersection in two points.

Note. The equation (A) gives the distances of the points of intersection P and Q from the point (α, β, γ), if l, m, n are direction cosines.

Exercises

1. Show that the locus of mid-points of chords of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

drawn parallel to the line

$$x/l=y/m=z/n$$

is the plane

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0.$$

[*Hint.* If (α, β, γ) be the middle point of any such chord

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

the two roots of the equation (A) are equal and opposite and as such their sum is zero.]

2. Find the locus of the chords of a cone which are bisected at a fixed point.

7·41. The tangent lines and tangent plane at a point.

Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

be *any* line through a point (α, β, γ) of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(ii)$$

so that

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0.$$

Thus one of the values of r given by the equation (A) of Art. 7·4 is zero and so one of the two points of intersection coincides with (α, β, γ). The second point of intersection will also coincide with (α, β, γ) if the second root of the same equation is also zero. This requires

$$l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0 \qquad \ldots(iii)$$

The line (i) corresponding to the set of values of l, m, n, satisfying the relation (iii) is *a tangent line* at (α, β, γ) to the cone (ii).

Eliminating l, m, n, between (i) and (iii), we obtain the locus of all the tangent lines through (α, β, γ), *viz.*,

$$(x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma)+(z-\gamma)(g\alpha+f\beta+c\gamma)=0$$

or

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)$$
$$=a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0,$$

which is a plane known as the **tangent plane.**

Clearly the tangent plane at any point passes through the vertex.

Cor. The tangent plane at *any* point $(k\alpha, k\beta, k\gamma)$ on the generator through the point (α, β, γ) is the same as the tangent plane at (α, β, γ).

Thus we see that the *tangent plane at any point on a cone touches it at all points of the generator through that point and we say that the plane touches the cone along the generator.*

Examples

1. *Show that*

$$x/l=y/m=z/n$$

is the line of intersection of the tangent planes to the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

along the lines in which it is cut by the plane

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0.$$

The tangent plane at any point (α, β, γ) of the given cone is

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0.$$

It will contain the line

$$x/l=y/m=z/n,$$

if

$$l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0,$$

i.e., $$\alpha(al+hm+gn)+\beta(hl+bm+fn)+\gamma(gl+fm+cn)=0.$$

Thus the point (α, β, γ) lies on the plane

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0.$$

Hence the result.

2. *Show that the locus of the line of intersection of tangent planes to the cone*

$$ax^2+by^2+cz^2=0$$

which touch along perpendicular generators is the cone

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.$$

Let the tangent planes along two perpendicular generators of the cone meet in the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \qquad \ldots(i)$$

Therefore, the equation of the plane containing the two generators is

$$alx+bmy+cnz=0. \qquad \ldots(ii)$$

Let λ, μ, ν, be the direction ratios of any one of the two generators so that we have

$$al\lambda+bm\mu+cn\nu=0, \qquad \ldots(iii)$$

$$a\lambda^2+b\mu^2+c\nu^2=0. \qquad \ldots(iv)$$

Eliminating ν from (iii) and (iv), we have

$$a(cn^2+al^2)\lambda^2+2ablm\lambda\mu+b(cn^2+bm^2)\mu^2=0.$$

If λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2, be the direction cosines of the two generators, we have

$$\frac{\lambda_1\lambda_2}{\mu_1\mu_2}=\frac{b(cn^2+bm^2)}{a(cn^2+al^2)}$$

or

$$\frac{\lambda_1\lambda_2}{(cn^2+bm^2)/a}=\frac{\mu_1\mu_2}{(cn^2+al^2)/b}.$$

Hence, by symmetry, we get

$$\frac{\lambda_1\lambda_2}{(cn^2+bm^2)/a}=\frac{\mu_1\mu_2}{(cn^2+al^2)/b}=\frac{\nu_1\nu_2}{(al^2+bm^2)/c}.$$

The generators being at right angle, we have

$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0,$$

i.e.,

$$\frac{cn^2+bm^2}{a}+\frac{cn^2+al^2}{b}+\frac{al^2+bm^2}{c}=0,$$

or

$$a^2(b+c)l^2+b^2(c+a)m^2+c^2(a+b)n^2=0. \qquad \ldots(v)$$

Eliminating $l : m : n$ from (i) and (v), we obtain

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0,$$

as the required locus.

7·42. Condition for tangency. *To find the condition that the plane*

$$lx+my+nz=0, \qquad \ldots(1)$$

should touch the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0. \qquad \ldots(2)$$

If (α, β, γ) be the point of contact, the tangent plane

$$x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0$$

thereat should be the same as the plane (1).

$$\therefore \quad \frac{a\alpha+h\beta+g\gamma}{l}=\frac{h\alpha+b\beta+f\gamma}{m}=\frac{g\alpha+f\beta+c\gamma}{n}=k. \text{ (say).}$$

Hence

$$a\alpha+h\beta+g\gamma-lk=0, \qquad \ldots (i)$$

$$h\alpha+b\beta+f\gamma-mk=0, \qquad \ldots (ii)$$

$$g\alpha+f\beta+c\gamma-nk=0. \qquad \ldots (iii)$$

Also, since (α, β, γ) lies on the plane (1), we have

$$l\alpha+m\beta+n\gamma=0. \qquad \ldots (iv)$$

Eliminating α, β, γ, k between (i), (ii), (iii), (iv), we obtain

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix} =0, \qquad \ldots(A)$$

as the required condition.

The determinant (A), on expansion, gives

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$

where A, B, C, F, G, H are, as usual, the co-factors of a, b, c, f, g, h respectively in the determinant

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

i.e., $A=bc-f^2,\ B=ca-g^2,\ C=ab-h^2$;

$F=gh-af,\ G=hf-bg,\ H=fg-ch.$

7·53. Reciprocal cones. *To find the locus of lines through the vertex of the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(1)$$

perpendicular to its tangent planes.

Let

$$lx+my+nz=0, \qquad \ldots(2)$$

be any tangent plane to the cone (1) so that we have

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0. \qquad \ldots(3)$$

The line through the vertex perpendicular to the tangent plane (2) is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \qquad \ldots(4)$$

Eliminating l, m, n between (3), (4), we get

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \qquad \ldots(5)$$

as the required locus which is again a quadric cone with its vertex at the origin.

If we now find the locus of lines through the origin perpendicular to the tangent planes to the cone (5), we have to substitute for A, B, C, F, G, H in its equation the corresponding co-factors in the determinant

$$\begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix}.$$

Since, we have, by actual multiplication,

$$BC-F^2=aD, \quad CA-G^2=bD, \quad AB-H^2=cD;$$
$$GH-AF=fD, \quad HF-BG=gD, \quad FG-CH=hD;$$

where

$$D\equiv abc+2fgh-af^2-bg^2-ch^2,$$

it follows that the required locus for the cone (5) is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

which is the same as (1).

The two cones (1) and (5) are, therefore, such that each is the locus of the normals drawn through the origin to the tangent planes to the other and they are, on this account, called *reciprocal cones*.

Cor. *The condition for the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \qquad \ldots(i)$$

to possess three mutually perpendicular tangent planes is

$$A+B+C=0.$$

The cone (i) will clearly possess three mutually perpendicular tangent planes, if its reciprocal cone

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0,$$

has three mutually perpendicular generators and this will be so if

$$A+B+C=0, \textit{ i.e., if } \mathbf{bc+ca+ab=f^2+g^2+h^2}.$$

Examples

1. *Show that the general equation of a cone which touches the three co-ordinate planes is*

$$\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0.$$

The reciprocal of a cone touching the three co-ordinate planes is a cone with three co-ordinate axes as three of its generators. Now,

the general equation of a cone through the three axes is

$$fyz+gzx+hxy=0.$$

Its reciprocal cone is

$$-f^2x^2-g^2y^2-h^2z^2+2ghyz+2hfzx+2fgxy=0,$$

or $$(fx+gy-hz)^2=4fgxy,$$

or $$fx+gy-hz=\pm 2\sqrt{fgxy},$$

or $$fx+gy\pm 2\sqrt{fgxy}=hz,$$

or $$(\sqrt{fx}\pm\sqrt{gy})^2=hz,$$

or $$\sqrt{fx}\pm\sqrt{gy}=\pm\sqrt{hz},$$

or $$\sqrt{fx}\pm\sqrt{gy}\pm\sqrt{hz}=0.$$

2. *Show that the locus of the line of intersection of perpendicular tangent planes to the cone*

$$ax^2+by^2+cz^2=0,$$

is the cone

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

Generators of the reciprocal cone corresponding to the *perpendicular* tangent planes of the original cone are themselves perpendicular. Also, the line of intersection of the perpendicular tangent planes is perpendicular to the corresponding generators of the reciprocal cone. Combining these two facts, we see that the given question is equivalent to determining the locus of normals through the origin to the planes which cut the reciprocal cone along perpendicular generators.

Equation of the reciprocal cone is

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0,$$

or

$$bcx^2+cay^2+abz^2=0. \qquad \ldots(i)$$

Let the plane

$$lx+my+nz=0 \qquad \ldots(ii)$$

cut the cone (*i*) along perpendicular generators. The condition for this, as may be easily obtained, is

$$a(b+c)l^2+b(c+a)m^2+c(a+b)n^2=0. \qquad \ldots(iii)$$

The equations of the normal to the plane (*ii*) are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \qquad \ldots(iv)$$

Eliminating l, m, n from (*iii*) and (*iv*), we obtain

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0,$$

as the required locus.

Exercises

1. Find the plane which touches the cone

$$x^2+2y^2-3z^2+2yz-5zx+3xy=0,$$

along the generator whose direction ratios are 1, 1, 1.

[*Ans.* $y=z$.

2. Prove that perpendiculars drawn from the origin to the tangent planes to the cone

$$ax^2+by^2+cz^2=0,$$

lie on the cone

$$x^2/a+y^2/b+z^2/c=0.$$

3. Prove that tangent planes to the cone

$$x^2-y^2+2z^2-3yz+4zx-5xy=0,$$

are perpendicular to the generators of the cone

$$17x^2+8y^2+29z^2+28yz-46zx-16xy=0.$$

4. Prove that the cones

$$ayz+bzx+cxy=0,\ (ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=0$$

are reciprocal.

5. Prove that the cones

$$ax^2+by^2+cz^2=0$$

and

$$x^2/a+y^2/b+z^2/c=0$$

are reciprocal. *(D.U. Hons. 1957)*

6. Show that a quadric cone can be found to touch any five planes which meet at a point provided no three of them intersect in a line.

Find the equation of the cone which touches the three co-ordinate planes and the planes

$$x+2y+3z=0,\ 2x+3y+4z=0.$$

$$[Ans.\ (x)^{\frac{1}{2}}+(-6y)^{\frac{1}{2}}+(6z)^{\frac{1}{2}}=0.$$

7. Show that a quadric cone can be found to touch any two sets of three mutually perpendicular planes which meet in a point.

8. Find the equation of the quadric cone which touches the three co-ordinate planes and the three mutually perpendicular planes

$$x-y+z=0,\ 2x+3y+z=0,\ 4x-y-5z=0.$$

$$[Ans.\ 64x^2+9y^2+25z^2-30yz-80zx+48xy=0.$$

7·5. Intersection of two cones with a common vertex. Sections of two cones, having a common vertex, by any plane are two coplanar conics which, in general, intersect in four points.

The four lines joining the common vertex to the four points of intersection of these two coplanar conics are the four common generators of the two cones.

Therefore *two cones with a common vertex intersect, in general, in four common generators.*

In case two cones with the same vertex have five common generators, they coincide.

If

$$S=0,\ S'=0,$$

be the equations of two cones with origin as the common vertex, then

$$S+kS'=0,$$

is clearly the general equation of a cone whose vertex is at the origin and which passes through the four common generators of the cones

$$S=0,\ S'=0$$

If k be so chosen that $S+kS'=0$ becomes the product of two linear factors, then the corresponding equations obtained by putting the linear factors equal to zero represent a pair of planes through the common generators.

Such values of k are the roots of the k-cubic equation

$$\begin{vmatrix} a+ka', & h+kh', & g+kg' \\ h+kh', & b+kb', & f+kf' \\ g+kg', & f+kf', & c+kc' \end{vmatrix} = 0.$$

The three values of k give the three pairs of planes through the four common generators.

Exercises

1. Find the equation of the cone which passes through the common generators of the cones

$$-2x^2+4y^2+z^2=0 \text{ and } 10xy-2yz+5zx=0$$

and the line with direction cosines proportional to 1, 2, 3.

[*Ans.* $2x^2-4y^2-z^2+10xy-2yz+5zx=0$.

2. Show that the equation of the cone through the intersection of the cones

$x^2-2y^2+3z^2-4yz+5zx-6xy=0$ and $2x^2-3y^2+4z^2-5yz+6zx-10xy=0$

and the line with direction cosines proportional to 1, 1, 1 is

$$y^2-2z^2+3yz-4zx+2xy=0.$$

3. Show that the plane $3x+2y-4z=0$ passes through a pair of common generators of the cones

$$27x^2+20y^2-32z^2=0 \text{ and } 2yz+zx-4xy=0.$$

Also show that the plane containing the other two generators is

$$9x+10y+8z=0.$$

4. Show that the plane $3x-2y-z=0$ cuts the cones

$$3yz-2zx+2xy=0 \text{ and } 21x^2-4y^2-5z^2=0$$

in the same pair of perpendicular lines.

Also show that the plane $7x+2y+5z=0$ contains the remaining two common generators.

5. Two cones are described with guiding curves

$$xz=a^2,\ y=0\ ;\ yz=b^2,\ x=0,$$

and with any vertex. Show that if their four common generators meet the plane $z=0$ in four concyclic points, the vertex lies on the surface

$$z(x^2+y^2)=a^2x+b^2y.$$

6. Find the conditions that the lines of section of the plane $lx+my+nz=0$ and the cones $fyz+gzx+hxy=0$, $ax^2+by^2+cz^2=0$ should be coincident.

$$\left[Ans.\ \frac{bn^2+cm^2}{fmn}=\frac{cl^2+an^2}{gnl}=\frac{am^2+bl^2}{hlm}.\right.$$

7·6. The right circular Cone.

7·61. Def. *A right circular cone is a surface generated by a line which passes through a fixed point, and makes a constant angle with a fixed line through the fixed point.*

The fixed point is called the *vertex*, the fixed line the *axis* and the fixed angle the *semi-vertical* angle of the cone.

The justification for the name right circular cone is contained in the result obtained below.

The section of a right circular cone by any plane perpendicular to its axis is a circle.

Let a plane perpendicular to the axis ON of the right circular cone with semi-vertical angle, α, meet it at N.

Let P be any point of the section. Since ON is perpendicular to the plane which contains the line NP, we have

$$ON \perp NP$$

$$\therefore \quad \frac{PN}{ON} = \tan \angle NOP = \tan \alpha,$$

or $$PN = ON \tan \alpha,$$

which is constant for every position of the point P of the section.

Hence the section is a circle with N as its centre.

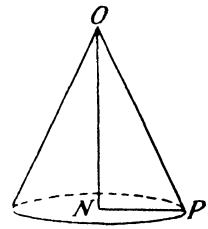

Fig. 24

7·62. Equation of right circular cone. *To find the equation of the right circular cone with its vertex at* (α, β, γ), *its axis the line*

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

and its semi-vertical angle θ.

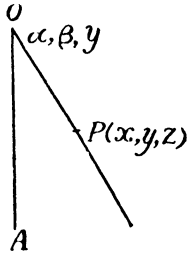

Fig. 25

Let O be the vertex, and, OA, the axis of the cone.

Any point $P(x, y, z)$ on the cone is such that the line joining it to the vertex O makes an angle θ with th axis OA.

Direction cosines of OP are, therefore, proportional to $x-\alpha, y-\beta, z-\gamma$.

$$\therefore \quad \cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]}}$$

The required equation of the cone, therefore, is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2 = (l^2+m^2+n^2)[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta.$$

Cor. 1. If the vertex be at the origin, the equation of the cone becomes

$$(lx+my+nz)^2 = (l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta.$$

Cor. 2. If the vertex be at the origin and axis be the z-axis, then taking

$$l=0,\ m=0,\ n=1$$

in the preceding Cor., we see that the equation of the cone becomes

$$z^2 = (x^2+y^2+z^2)\cos^2\theta, \quad i.e., \quad x^2+y^2 = z^2\tan^2\theta. \qquad \ldots(1)$$

Cor. 3. *The semi-vertical angle of a right circular cone having sets of three mutually perpendicular generators is*

$$\tan^{-1}\sqrt{2},$$

for, the sum of the co-efficients of x^2, y^2, z^2 in the equation of such a cone must be zero and this means that

$$1+1-\tan^2\theta = 0 \qquad \text{[Refer (1), Cor. 2.]}$$

i.e., $$\theta = \tan^{-1}\sqrt{2}.$$

Cor. 4. *The semi-vertical angle of a right circular cone having sets of three mutually perpendicular tangent planes is*

$$\tan^{-1}\sqrt{\tfrac{1}{2}},$$

for by Cor. to § 7·43, this will be so if [Refer (1), Cor. 2]

$$1-\tan^2\theta-\tan^2\theta=0$$

i.e., $$\tan\theta=\sqrt{\tfrac{1}{2}}.$$

or $$\theta=\tan^{-1}\sqrt{\tfrac{1}{2}}.$$

Exercises

1. Find the equation of the right circular cone with its vertex at the origin, axis along Z-axis and semi-vertical angle α.

[*Ans.* $x^2+y^2=z^2\tan^2\alpha$.

2. Show that the equation of the right circular cone with vertex (2, 3, 1), axis parallel to the line $-x=y/2=z$ and one of its generators having direction cosines proportional to (1, −1, 1) is

$$x^2-8y^2+z^2+12xy-12yz+6zx-46x+36y+22z-19=0.$$

3. Find the equation of the circular cone which passes through the point (1, 1, 2) and has its vertex at the origin and axis the line

$$x/2=-y/4=z/3.$$

[*Ans.* $4x^2+40y^2+19z^2-48xy-72yz+36zx=0$.

4. Find the equations of the circular cones which contain the three co-ordinate axes as generators.

[*Ans.* $yz\pm zx\pm xy=0$.

5. Lines are drawn through the origin with direction cosines proportional to (1, 2, 2), (2, 3, 6), (3, 4, 12). Show that the axis of the right circular cone through them has direction cosines

$$(-\sqrt{\tfrac{1}{3}},\ \sqrt{\tfrac{1}{3}},\ \sqrt{\tfrac{1}{3}}),$$

and that the semi-vertical angle of the cone is $\cos^{-1}(1/\sqrt{3})$.

Obtain the equation of the cone also.

[*Ans.* $xy-yz+zx=0$.

6. Find the equation of the right circular cone generated by straight lines drawn from the origin to cut the circle through the three points

(1, 2, 2), (2, 1, −2) and (2, −2, 1).

[*Ans.* $8x^2-4y^2-4z^2+5xy+yz+5zx=0$.

7. If α is the semi-vertical angle of the right circular cone which passes through the lines Oy, Oz, $x=y=z$, show that

$$\cos\alpha=(9-4\sqrt{3})^{-\frac{1}{2}}.$$

The Cylinder

7·7. *A* **cylinder** *is a surface generated by a straight line which is always parallel to a fixed line and is subject to one more condition ; for instance, it may intersect a given curve or touch a given surface.*

The given curve is called the *guiding curve.*

7·71. Equation of a cylinder. *To find the equation to the cylinder whose generators intersect the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0 \qquad \ldots(i)$$

and are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \qquad \ldots(ii)$$

Let (α, β, γ) be *any* point on the cylinder so that the equations of the generator through it are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(iii)$$

As in § 7·12, the line (*iii*) will intersect the conic (*i*), if

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2$$
$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0.$$

But this is the condition that the point (α, β, γ) should lie on the surface

$$a\left(x-\frac{lz}{n}\right)^2+2h\left(x-\frac{lz}{n}\right)\left(y-\frac{mz}{n}\right)+b\left(y-\frac{mz}{n}\right)^2+$$
$$2g\left(x-\frac{lz}{n}\right)+2f\left(y-\frac{mz}{n}\right)+c=0,$$

or $a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2+$
$$2gn(nx-lz)+2fn(ny-mz)+cn^2=0$$

which is, therefore, the required equation of the cylinder.

Cor. If the generators be parallel to Z-axis so that

$$l=0=m \text{ and } n=1,$$

the equation of the cylinder becomes

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

as is already known to the reader.

Exercises

1. Find the equation of the cylinder whose generators are parallel to

$$x=-\tfrac{1}{2}y=\tfrac{1}{3}z$$

and whose guiding curve is the ellipse

$$x^2+2y^2=1,\ z=3.$$

[*Ans.* $3(x^2+2y^2+z^2)+2(4yz-zx)+6(x-4y-3z)+24=0.$

2. Find the equation of the quadric cylinder whose generators intersect the curve $ax^2+by^2=2z$, $lx+my+nz=p$ and are parallel to Z-axis.

[Eliminate z from the two equations,]

[*Ans.* $n(ax^2+by^2)+2lx+2my-2p=0.$

3. Find the equation of the quadric cylinder with generators parallel to X-axis and passing through the curve

$$ax^2+by^2+cz^2=1,\ lx+my+nz=p.$$

[*Ans.* $(bl^2+am^2)y^2+2mnayz+(cl^2+an^2)z^2-2apmy-2apnz+(ap^2-l^2)=0.$

4. Show that the equation of the tangent plane at any point (α, β, γ) of the cylinder

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is

$$x(a\alpha+h\beta+g)+y(h\alpha+b\beta+f)+(g\alpha+f\beta+c)=0,$$

and that it touches the cylinder at all points of the generator through the point.

7·72. Enveloping Cylinder. *To find the equation to the cylinder whose generators touch the sphere*

$$x^2+y^2+z^2=a^2, \qquad \ldots(i)$$

and are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \qquad \ldots(ii)$$

Let (α, β, γ) be *any* point on the cylinder so that the equations of the generator through it are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(iii)$$

The line (*iii*) will touch the sphere (*i*) if

$$(l\alpha+m\beta+n\gamma)^2=(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2).$$

But this is the condition that the point (α, β, γ) should lie on the surface

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2),$$

which is, therefore, the required equation of the cylinder and isknown as *an enveloping cylinder of the sphere* (*i*).

Ex. Find the enveloping cylinder of the sphere

$$x^2+y^2+z^2-2x+4y=1,$$

having its generators parallel to the line

$$x=y=z.$$

[*Ans.* $x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$.

7·8. The Right Circular Cylinder.

7·81. *A right circular cylinder is a surface generated by a line which intersects a fixed circle, called the guiding circle, and is perpendicular to its plane.*

The normal to the plane of the guiding circle through its centre is called the *axis* of the cylinder.

Section of a right circular cylinder by any plane perpendicular to its axis is called a *normal section.*

Clearly all the normal sections are circles having the same radius which is also called the radius of the cylinder. *The length of the perpendicular from any point on a right circular cylinder to its axis is equal to its radius.*

7·82. Equation of a Right Circular Cylinder. *To find the equation of the right circular cylinder whose axis is the line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

and whose radius is r.

Let (x, y, z) be any point on the cylinder. Equating the perpendicular distance of the point from the axis to the radius r, we get

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2}=r^2,$$

which is the required equation of the cylinder.

Example

Find the equation of a right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction cosines proportional to (2, −3, 6). (*P. U. 1940*)

The axis of the right circular cylinder is

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6} \text{ or } \frac{x-1}{2/7}=\frac{y-2}{-3/7}=\frac{z-3}{6/7}.$$

Let (f, g, h) be *any* point of the cylinder. The square of the distance of the point (f, g, h) from the axis is

$$(f-1)^2+(g-2)^2+(h-3)^2-[\tfrac{2}{7}(f-1)-\tfrac{3}{7}(g-2)+\tfrac{6}{7}(h-3)]^2.$$

Equating it to the square of the radius 2, we see that the point (f, g, h) satisfies the equation

$$45f^2+40g^2+13h^2+36gh-24hf+12fg-42f-280g-126h+294=0$$

so that the required equation is

$$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0.$$

Exercises

1. Find the equation of the right circular cylinder of radius 2 whose axis is the line

$$(x-1)/2=(y-2)=(z-3)/2.$$

[*Ans.* $5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$.

2. The axis of a right circular cylinder of radius 2 is

$$\frac{x-1}{2}=\frac{y}{3}=\frac{z-3}{1};$$

show that its equation is

$$10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0.$$

3. Find the equation of the circular cylinder whose guiding circle is

$$x^2+y^2+z^2-9=0,\ x-y+z=3.$$

[*Ans.* $x^2+y^2+z^2+xy+yz-zx=9$,

[*Hint.* Show that the radius of the circle is $\sqrt{6}$ and the axis of the cylinder is $x=-y=z$.]

4. Obtain the equation of the right circular cylinder described on the circle through the three points (1, 0, 0), (0, 1, 0), (0, 0, 1) as guiding circle.

[*Ans.* $x^2+y^2+z^2-xy-yz-zx=1$.

Examples

1. *Find the angle between the lines in which the plane*

$$ux+vy+wz=0,$$

cuts the cone

$$ax^2+by^2+cz^2=0.$$

Let l, m, n be the direction cosines of any one of the two lines so that we have

$$ul+vm+wn=0, \qquad \ldots(i)$$

$$al^2+bm^2+cn^2=0. \qquad \ldots(ii)$$

Eliminating l from (i) and (ii), we obtain

$$(av^2+bu^2)m^2+2avwmn+(aw^2+cu^2)n^2=0,$$

or

$$(av^2+bu^2)\left(\frac{m}{n}\right)^2+2avw\left(\frac{m}{n}\right)+(aw^2+cu^2)=0. \qquad \ldots(iii)$$

Let l_1, m_1, n_1 ; l_2, m_2, n_2, be the direction cosines of the two lines separately. From (*iii*), we get

$$\frac{m_1}{n_1}\cdot\frac{m_2}{n_2}=\frac{aw^2+cu^2}{av^2+bu^2},$$

and

$$\frac{m_1}{n_1}+\frac{m_2}{n_2}=-\frac{2avw}{av^2+bu^2}.$$

Thus

$$\begin{aligned}\frac{m_1m_2}{aw^2+cu^2}&=\frac{n_1n_2}{av^2+bu^2}=\frac{m_1n_2+m_2n_1}{-2avw}\\&=\pm\frac{\sqrt{[(m_1n_2+m_2n_1)^2-4m_1m_2n_1n_2]}}{\sqrt{[4a^2v^2w^2-4(aw^2+cu^2)(av^2+bu^2)]}}\\&=\pm\frac{m_1n_2-m_2n_1}{2u\sqrt{[-(u^2bc+v^2ca+w^2ab)]}}.\end{aligned}$$

From symmetry, each of these expressions is equal to

$$\begin{aligned}\frac{l_1l_2}{bw^2+cu^2}&=\pm\frac{n_1l_2-n_2l_1}{2v\sqrt{[-(u^2bc+v^2ca+w^2ab)]}}\\&=\pm\frac{l_1m_2-l_2m_1}{2w\sqrt{[-(u^2bc+v^2ca+w^2ab)]}}=k,\ \text{(say)}\end{aligned}$$

If θ be the angle between the two lines, we have

$$\begin{aligned}\tan\theta&=\frac{\sqrt{[\Sigma(m_1n_2-m_2n_1)^2]}}{\Sigma l_1l_2}\\&=\pm\frac{\sqrt{[-4(u^2+v^2+w^2)(u^2bc+v^2ca+w^2ab)]}}{a(v^2+w^2)+b(w^2+u^2)+c(u^2+v^2)}.\end{aligned}$$

2. *P, Q are the points of intersection of the line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

with the cone

$$ax^2+by^2+cz^2=0.$$

Show that the sphere described on PQ as diameter will pass through the vertex of the cone, if

$$a(\mu^2+\nu^2)+b(\nu^2+\lambda^2)+c(\lambda^2+\mu^2)=0,$$

where

$$\lambda=\beta n-\gamma l,\ \mu=\gamma l-\alpha n,\ \nu=\alpha m-\beta l.$$

Any point $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$ on the line will lie on the cone if

$$(al^2+bm^2+cn^2)r^2+2(al\alpha+bm\beta+cn\gamma)r+(a\alpha^2+b\beta^2+c\gamma^2)=0. \quad \ldots(i)$$

Let r_1, r_2 be the roots of this r-quadratic. Therefore, the points P, Q are

$$(lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma),\ (lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma).$$

Thus

$$\left.\begin{aligned}r_1+r_2&=\frac{-2(al\alpha+bm\beta+cn\gamma)}{al^2+bm^2+cn^2}=\frac{-2\Sigma al\alpha}{\Sigma al^2},\\r_1r_2&=\frac{a\alpha^2+b\beta^2+c\gamma^2}{al^2+bm^2+cn^2}=\frac{\Sigma a\alpha^2}{\Sigma al^2}.\end{aligned}\right\}\quad\ldots(ii)$$

The sphere on PQ as diameter is

$$\Sigma(x-\overline{lr_1+\alpha})(x-\overline{lr_2+\alpha})=0,$$

or $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-$
$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)](r_1+r_2)+(l^2+m^2+n^2)r_1r_2=0,$

which, with the help of (i), becomes

$$\Sigma(x-a)^2\Sigma al^2+2\Sigma l(x-\alpha)\Sigma al\alpha+\Sigma l^2\Sigma a\alpha^2=0.$$

It will pass through (0, 0, 0), if

$$(\alpha^2+\beta^2+\gamma^2)(al^2+bm^2+cn^2)-2(l\alpha+m\beta+n\gamma)(al\alpha+bm\beta+cn\gamma)+$$
$$(l^2+m^2+n^2)(a\alpha^2+b\beta^2+c\gamma^2)=0,$$

or $$\Sigma a[(l\gamma-\alpha n)^2+(\alpha m-\beta l)^2]=0,$$

i.e., $$\Sigma a(\mu^2+\nu^2)=0.$$

3. *A sphere passes through the circle*

$$z=0,\ x^2+y^2=a^2.$$

Prove that the locus of the extremities of its diameter parallel to X-axis is the rectangular hyperbola

$$y=0,\ x^2-z^2=a^2. \qquad (B.U.)$$

The equation of the general sphere through the given circle is

$$x^2+y^2+z^2+\lambda z=a^2\ ; \qquad \ldots(i)$$

λ being the parameter.

Its centre is $(0, 0, -\frac{1}{2}\lambda)$. Therefore, the equations of its diameter parallel to X-axis are

$$y=0,\ z=-\tfrac{1}{2}\lambda. \qquad \ldots(ii)$$

Eliminating λ between (i) and (ii), we get

$$y=0,\ x^2-z^2=a^2$$

as the required locus which is clearly a rectangular hyperbola.

Revision Exercises II

1. Show that the plane $x+2y-z=4$ cuts the sphere

$$x^2+y^2+z^2-x+z=2$$

in a circle of radius unity and find the equation of the sphere which has this circle as one of its great circles.

[*Ans.* $x^2+y^2+z^2-2x-2y+2z+2=0$.

2. Find the equation of the sphere which passes through the point (2, 3, 6) and the feet of the perpendiculars from this point on the co-ordinate planes.

Also find the equations of tangent planes to the sphere which are parallel to the plane $2x+2y+z=0$; and the co-ordinates of their points of contact.

[*Ans.* $x^2+y^2+z^2-2x-3y-6z=0$; $4x+4y+2z+5=0$; $4x+4y+2z-37=0$, $(-\frac{4}{3}, -\frac{5}{6}, \frac{11}{6})$; $(\frac{10}{3}, \frac{23}{6}, \frac{25}{6})$.

3. Show that all the spheres that can be drawn through the origin and each set of points where planes parallel to the plane

$$x/a+y/b+z/c=0$$

cut the co-ordinate axes form a system of spheres which are cut orthogonally by the spheres

$$x^2+y^2+z^2+2fx+2gy+2hz=0 \text{ if } af+bg+ch=0. \qquad (M.T.)$$

4. Find the equations of the lines passing through the point (1, 1, 1), tangent to the sphere $x^2+y^2+z^2=2$ and parallel to the plane

$$4x+3y-z=0.$$

[*Ans.* $3-2x=y=z$; $2(x-1)=3(1-y)=z-1$.

5. Obtain the equations of the planes passing through the point (3, 0, 3), tangent to the sphere $x^2+y^2+z^2=9$ and parallel to the line

$$x=2y=-z.$$

[*Ans.* $x+2y+2z=9$; $2x-2y+z=9$.

6. Find the equations of the spheres which touch the planes $x=0$, $y=0$, $z=0$, lie on the positive sides of these planes and are cut orthogonally by the sphere

$$x^2+y^2+z^2-4x-6y+2z+6=0.$$

[*Ans.* $x^2+y^2+z^2-2(x+y+z)+2=0$, $x^2+y^2+z^2-6(x+y+z)+18=0$.

7. Find the equations of the spheres that pass through the points (−3, 4, 1), (−1, 0, −3), (0, 3, −3) and touch the line $x=y$, $z=0$.

[*Ans.* $x^2+y^2+z^2+4x-4y+2z=0$,

$$6(x^2+y^2+z^2)+402x-150y+327z+1323=0.$$

8. Show that the line $(x-a)/l=(y-b)/m=(z-c)/n$ is touched by two spheres, each of which passes through the points (0, 0, 0), (2*a*, 0, 0), (0, 2*b*, 0). Show further that the distance between the centres of the spheres is

$$2[c^2-n^2(a^2+b^2+c^2)]^{\frac{1}{2}}/n^2.$$

9. Find the equations of the tangent to the circle through the three points (−3, 0, 1), (5, 1, −2), (0, 4, 2) at the point (−3, 0, 1).

[*Ans.* $\frac{x+3}{7}=\frac{y}{135}=\frac{z-1}{76}$.

10. Find the equation of the sphere inscribed in the tetrahedron formed by the planes whose equations are

$$y+z=0,\ z+x=0,\ x+y=0,\ x+y+z=1.$$

[*Ans.* $x^2+y^2+z^2-2a(x+y+z)+a^2=0$, where $(3+\sqrt{6})a=1$.

11. If A, A' are points where the lines $y=mx$, $z=c$; $y=-mx$, $z=-c$, meet the shortest distance between them and P, P' are the points, one on each of these lines, such that the sphere on PP' as diameter cuts orthogonally the sphere on AA' as diameter, show that PP' lies on the surface

$$(1-m^2)(y^2-m^2x^2)=2m^2(z^2-c^2). \qquad (B.U.)$$

12. POP' is a variable diameter of the hyperbola $x^2/a^2-y^2/b^2=1$, $z=0$, and a circle is described in the plane $PP'ZZ'$ on PP' as diameter. Prove that as $PP'O$ varies, the circle generates the surface

$$(x^2+y^2+z^2)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=x^2+y^2.$$

13. A variable plane is parallel to a given plane $x/a+y/b+z/c=0$ and meets the axes in A, B, C. Prove that the circle A, B, C lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

(*D.U. Hons., 1959 ; B.U.*)

14. A straight line whose equations in one position are

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$$

is rotated about the axis of Z ; prove that the surface generated is

$$n^2(x^2+y^2)=(lz+na-lc)^2+(mz+nb-mc)^2.$$

15. Find the equation of the system of spheres which touch Z-axis at the origin and pass through a fixed point (a, b, c) ; show that all these spheres pass through a fixed circle.

16. Find the equations of the two spheres whose centres lie in the positive octant and which touch the planes

$$x=0,\ y=0,\ z=0,\ x+2y+2z=8a.$$

[*Ans.* (*i*) $x^2+y^2+z^2-2a(x+y+z)+2a^2=0$. (*ii*) $x^2+y^2+z^2-8a(x+y+z)+32a^2=0$.

17. Find the equation of the sphere which cuts orthogonally each of the four spheres

$$x^2+y^2+z^2=a^2+b^2+c^2, \qquad x^2+y^2+z^2+2ax=a^2,$$
$$x^2+y^2+z^2+2by=b^2, \qquad x^2+y^2+z^2+2cz=c^2. \qquad (M.T.)$$

$$[Ans.\quad x^2+y^2+z^2+\frac{b^2+c^2}{a}x+\frac{c^2+a^2}{b}y+\frac{a^2+b^2}{c}z+(a^2+b^2+c^2)=0.$$

18. Show that the locus of the centre of a variable sphere which cuts a fixed sphere $S=0$ orthogonally and is cut by another sphere $S'=0$ along a great circle is the sphere $S+S'=0$.

19. Prove that the locus of the centre of a variable sphere which cuts each of two given spheres in great circles is a plane perpendicular to their line of centres.

20. Find the locus of the centres of spheres which touch the two lines

$$y=\pm mx,\ z=\pm c.$$

[*Ans.* $mxy+cz(1+m)^2=0$.

21. A sphere of radius R passes through the origin; show that the extremities of the diameter parallel to the X-axis lie one on each of the spheres

$$x^2+y^2+z^2\pm 2Rx=0. \qquad (L.U.\ 1907)$$

22. Show that the cone $yz+zx+xy=0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles, and find their area. [*Ans.* $\frac{2}{3}\pi a^2$.

23. (*i*) Find the angles between the lines in which the plane

$$ux+vy+wz=0,$$

cuts the cone

$$ayz+bzx+cxy=0.$$

$$[Ans.\quad \tan^{-1}\frac{[\sum u^2a^2-2\sum bcvw][\sum u^2]^{\frac{1}{2}}}{\sum avw}$$

(*ii*) Show that the plane

$$ax+by+cz=0,$$

cuts the cone

$$yz+zx+xy=0$$

in two lines inclined at an angle

$$\tan^{-1}\left[\frac{\{(a^2+b^2+c^2)(a^2+b^2+c^2-2ab-2bc-2ca)\}^{\frac{1}{2}}}{bc+ca+ab}\right].$$

(*D.U. Hons. 1958*)

24. Show that the angle between the lines given by

$$x+y+z=0,\ ayz+bzx+cxy=0,$$

is

$\frac{1}{2}\pi$ if $a+b+c=0$ and $\frac{1}{3}\pi$ if $a^{-1}+b^{-1}+c^{-1}=0$.

(*D.U. Hons., 1959*)

25. Find the equation of the cone generated by straight lines drawn from the origin to cut the circle through the three points (1, 0, 0), (0, 2, 0), (2, 1, 1) and prove that acute angle between the two lines in which the plane $x=2y$ cuts the cone is $\cos^{-1}\sqrt{(5/14)}$. (*M.T.*)

[*Ans.* $8z^2-zx-5xy+4yz=0$.

26. A cone has for its guiding curve the circle

$$x^2+y^2+2ax+2by=0,\ z=0$$

and passes through a fixed point $(0, 0, c)$. If the section of the cone by the plane $x=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle

$$x^2+y^2+z^2+2ax+2by=0,\ 2ax+2by+cz=0.$$

27. Planes through X-axis and Y-axis include an angle α ; show that locus of their lines of intersection is the cone

$$z^2(x^2+y^2+z^2)=x^2y^2\tan^2\alpha.$$

28. Prove that the straight lines which cut two given skew lines such that the length intercepted is constant, are parallel to the generators of a circular cone whose axis lies along the line of shortest distance between the given lines.

29. Show that the plane $z=a$ meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$ in a conic which has a focus at the point $(0, 0, a)$. *(P. U. 1938)*

30. A point O is at a constant distance $2a$ from the origin and points P, Q, R are taken on the axes in such a way that OP, OQ, OR are mutually perpendicular. Prove that the plane PQR always touches a fixed sphere of radius a. *(M.T.)*

31. Show that, in general, two spheres may be drawn to contain a given circle and to touch a given plane. If the circle lies in the plane $z=0$, and has a given radius a, and if the plane is $x\cos\theta+z\sin\theta=0$, show that if the distance between the centres of the two spheres is constant and equal to $2c$, the locus of the centre of the circle is the pair of lines

$$x=\pm\sqrt{(a^2+c^2\cos^2\theta)},\ z=0. \qquad (P.\ U.\ 1949)$$

32. Find the equation of the right circular cone with the vertex $(1, -2, -1)$, semi-vertical angle $60°$ and the line

$$\frac{x+1}{3}=\frac{y+2}{-4}=\frac{z+1}{5}$$

as its axis. Prove that the plane $3x-4y+5z=56$ cuts it in a circle. Find its centre and radius. Find the equation of the right circular cylinder with the above circle as its base. *(P.U. 1948)*

[*Ans.* $7x^2-7y^2-25z^2+48xy+80yz-60zx+22x+4y+170z+78=0$

centre $(4, -6, 4)$; radius $5\sqrt{6}$.

$41x^2+34y^2+25z^2+24xy+40yz-30zx-64x+152y+160z-7236=0$.

33. At what angle does the sphere

$$x^2+y^2+z^2-2x-4y-6z+10=0$$

intersect the sphere which has the points $(1, 2, -3)$ and $(5, 0, 1)$ as extremities of a diameter. Find the equation to the sphere through the point $(0, 0, 0)$ coaxal with the above two spheres. *(P.U. 1948)*

[*Ans.* $\cos^{-1}(-\frac{2}{3})$; $2(x^2+y^2+z^2)-14x-3y+8z=0$.

34. A line with direction ratios $l : m : n$ is drawn through the fixed point $(0, 0, a)$ to touch the sphere

$$x^2+y^2+z^2-2ax=0.$$

Prove that

$$n^2+2nl=0.$$

Find the co-ordinates of the point P in which this line meets the plane $z=0$ and prove that as the line varies, P traces out the parabola

$$y^2=2ax,\ z=0.$$

APPENDIX

HOMOGENEOUS CARTESIAN CO-ORDINATES

ELEMENTS AT INFINITY

A. 1. Let X, Y, Z be the cartesian co-ordinates of any point P and let x, y, z, w be *any* four numbers such that

$$X=\frac{x}{w},\ Y=\frac{y}{w},\ Z=\frac{z}{w}, \qquad (w\neq 0).$$

Then we say that x, y, z, w are the homogeneous cartesian co-ordinates (or simply homogeneous co-ordinates for the purposes of this book), of the point P. Also, if x, y, z, w are the homogeneous co-ordinates of a point P, then the four numbers kx, ky, kz, kw, $(k\neq 0)$ which are proportional to x, y, z, w are also the homogeneous co-ordinates of the same point, for,

$$\frac{kx}{kw}=\frac{x}{w}=X, \text{ etc.}$$

In particular, $(x, y, z, 1)$, are the homogeneous co-ordinates of the point whose ordinary co-ordinates are (x, y, z).

Conversely, if the homogeneous co-ordinates of a point are (x, y, z, w), then its ordinary co-ordinates are

$$(x/w,\ y/w,\ z/w).$$

A. 11. Equation of a plane in Homogeneous co-ordinates. In the ordinary cartesian equation,

$$AX+BY+CZ+D=0,$$

of a plane, if we change X, Y, Z to $x/w,\ y/w,\ z/w$, respectively, we obtain

$$Ax+By+Cz+Dw=0,$$

which is the general equation of a plane in homogenous cartesian co-ordinates ; x, y, z, w, being the current co-ordinates.

A. 12. Equation of a line in Homogeneous co-ordinates. As above we can easily see that, in homogeneous cartesian co-ordinates, the equations of the straight line through (x', y', z', w') with direction ratios l, m, n are

$$\frac{xw'-wx'}{l}=\frac{yw'-wy'}{m}=\frac{zw'-wz'}{n}.$$

Also, we may easily see that the equations of the line through (x_1, y_1, z_1, w_1), and (x_2, y_2, z_2, w_2) are

$$\frac{xw_1-x_1w}{x_1w_2-x_2w_1}=\frac{yw_1-y_1w}{y_1w_2-y_2w_1}=\frac{zw_1-z_1w}{z_1w_2-z_2w_1}.$$

Ex. Show that any point on the line joining
(x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2)
is
$$(x_1+tx_2, y_1+ty_2, z_1+tz_2, w_1+tw_2) ;$$
t being the parameter.

A. 2. Elements at infinity. Let (x, y, z, w) be the homogeneous co-ordinates of any point. If x, y, z are not all zero and $w \to 0$ then, one at least of the three ordinary co-ordinates x/w, y/w, z/w, tends to infinity. We find it convenient to express this idea by saying that (x, y, z, w), when $w=0$ and x, y, z *are not all* 0, *is a point at infinity.* The aggregate of the points $(x, y, z, 0)$ where x, y, z take up different sets of values, not all zero, is the aggregate of the points at infinity. The equation of the locus of points at infinity is
$$w=0,$$
which being of the first degree, we *say* that the locus of the points at infinity in space is a plane and call it the **plane at infinity.**

A. 21. *Two parallel lines meet at a point at infinity.* Consider the two parallel lines
$$\frac{xw_1-x_1w}{l}=\frac{yw_1-y_1w}{m}=\frac{zw_1-z_1w}{n}, \qquad \ldots(i)$$
$$\frac{xw_2-x_2w}{l}=\frac{yw_2-y_2w}{m}=\frac{zw_2-z_2w}{n}. \qquad \ldots(ii)$$
Putting $w=0$, we obtain
$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$
for both (i), (ii) so that we see that both the lines meet at the point at infinity $(l, m, n, 0)$.

It is useful to remember that $(l, m, n, 0)$ *is the point at infinity on every line with direction ratios* l, m, n.

A. 22. Line at infinity on a plane. The aggregate of points (x, y, z, w) which satisfy the equation
$$w=0 \qquad \ldots(iii)$$
of the plane at infinity, and the equation
$$Ax+By+Cz+Dw=0 \qquad \ldots(iv)$$
of any arbitrary plane, is said to be the **line at infinity,** on the plane (iv).

Thus, for the line at infinity on the plane (iv), we have the equations
$$Ax+By+Cz=0, \; w=0.$$

A. 23. *Two parallel planes have a common line at infinity.* Putting $w=0$ in the equations of the two parallel planes
$$Ax+By+Cz+Dw=0,$$
$$Ax+By+Cz+D'w=0,$$
we see that they both contain the same line at infinity, *viz.*,
$$Ax+By+Cz=0, \; w=0.$$

Note. The importance of the notions of 'points at infinity' and 'lines at infinity' lies in the fact that in certain cases we can replace directions of lines and orientations of planes by points and lines lying on the plane at infinity.

A. 3. Illustrations.

1. *Find the equation of the plane through the points*

$$(1, 0, -1), (3, 2, 2)$$

and parallel to the line

$$\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}.$$

In the notation of homogeneous co-ordinates, we are required to find the plane through the three points

$$(1, 0, -1, 1), (3, 2, 2, 1), (1, -2, 3, 0);$$

the last one being the point at infinity on the given line.

The required equation is

$$Ax+By+Cz+Dw=0, \qquad \ldots(1)$$

where A, B, C, D are to be determined from the three simultaneous linear equations

$$A-C+D=0,\ 3A+2B+2C+D=0,\ A-2B+3C=0.$$

Solving these for $A : B : C : D$ and substituting the values in (1), we see that the required equation is

$$4x-y-2z-6w=0$$

i.e.,

$$4x-y-2z-6=0,$$

in the notation of ordinary co-ordinates.

2. *Find the condition for the lines*

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

to be coplanar.

In the notation of homogeneous co-ordinates, we see that the first line is the join of the points

$$(x_1, y_1, z_1, 1) \text{ and } (l_1, m_1, n_1, 0)$$

and the second is the join of

$$(x_2, y_2, z_2, 1) \text{ and } (l_2, m_2, n_2, 0).$$

The necessary and sufficient condition for the two lines to be coplanar is that these four points be co-planar for which we have the condition

$$\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ l_1, & m_1, & n_1, & 0 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix}=0,$$

i.e.,
$$\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2, & 0 \\ x_2, & y_2, & z_2, & 1 \\ l_1, & m_1, & n_1, & 0 \\ l_2, & m_2, & n_2, & 0 \end{vmatrix} = 0,$$

or
$$\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0$$

which is the same as obtained in § 3·4, pp. 44-45.

3. Regarding a cylinder as a cone whose vertex is a point at infinity, we can deduce the equation of the cylinder whose guiding curve is

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0 \qquad \ldots(1)$$

and the direction ratios of whose generators are l, m, n, from the equation of the cone whose guiding curve is (1) and whose vertex is (α, β, γ).

A. 4. Sphere in Homogeneous co-ordinates. Changing x, y, z to $x/w, y/w, z/w$ respectively in the general equation of a sphere, we see that

$$x^2+y^2+z^2+2fxw+2gyw+2hzw+dw^2=0, \qquad \ldots(1)$$

is the general equation of a sphere in Homogeneous cartesian co-ordinates.

A. 41. Section of a sphere by the plane at infinity. Putting $w=0$ in (1), we see that the section of (1) by the plane at infinity is the curve

$$x^2+y^2+z^2=0,\ w=0. \qquad \ldots(2)$$

From the fact that the equations (2) do not involve the arbitrary constants f, g, h, d, we deduce that *every sphere meets the plane at infinity in the same curve.* The plane curve (2) which lies on every sphere is known as "*The absolute circle*," or the "*Circle at infinity.*"

We shall now show that

Every surface of the second degree which contains the circle at infinity is a sphere.

To prove this, we consider the general second degree equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2pxw+2qyw+2rzw+dw^2=0. \qquad \ldots(3)$$

Putting $w=0$, we obtain

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

which will be identical with $x^2+y^2+z^2=0$, if, and only if,

$$a=b=c \text{ and } f=0,\ g=0,\ h=0,$$

i.e. if (3) is a sphere.

A. 5. Relationship of perpendicularity in terms of conjugacy.

Let l, m, n, and l', m', n', be the direction ratios of two lines. The points at infinity

$$(l, m, n, 0), (l', m', n', 0)$$

on these two lines will be conjugate with regard to the circle at infinity, if

$$ll'+mm'+nn'=0$$

i.e., if the two lines are perpendicular.

Thus, we see that two lines are perpendicular if the points at infinity on them are conjugate with regard to the circle at infinity.

The lines at infinity

$$ax+by+cz=0=w\ ;\ a'x+b'y+c'z=0=w,$$

on the two planes

$$ax+by+cz+dw=0,\ a'x+b'y+c'z+d'w=0,$$

will be conjugate for the circle at infinity, if

$$aa'+bb'+cc'=0,$$

i.e., if the two planes are perpendicular.

Thus we see that *two planes are perpendicular if the lines at infinity on them are conjugate with respect to the circle at infinity.*

It may also be easily shown in a similar manner *that a line is perpendicular to a plane if the point at infinity on the line is the pole of the line at infinity on the plane with regard to the circle at infinity.*

CHAPTER VIII

THE CONICOID

The general equation of the second degree

8·1. The locus of the general equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

of the second degree in x, y, z is called a **conicoid** or a **quadric.**

It is easy to show that every straight line meets a surface whose equation is of the second degree in two points and consequently every plane section of such a surface is a conic. This property justifies the name "Conicoid" as applied to such a surface.

The general equation of second degree contains *nine* effective constants and, therefore, a conicoid can be determined to satisfy nine conditions each of which gives rise to one relation between the constants, *e.g.*, a conicoid can be determined so as to pass through *nine* given points no four of which are coplanar.

The general equation of the second degree can, by transformation of co-ordinate axes, be reduced to any one of the following forms ; the actual reduction being given in Chapter XI. (The name of the particular surface which is the locus of the equation is written along with it.)

1. $x^2/a^2+y^2/b^2+z^2/c^2=1$, Ellipsoid.
2. $x^2/a^2+y^2/b^2+z^2/c^2=-1$, Imaginary ellipsoid.
3. $x^2/a^2+y^2/b^2-z^2/c^2=1$, Hyperboloid of one sheet.
4. $x^2/a^2-y^2/b^2-z^2/c^2=1$, Hyperboloid of two sheets.
5. $x^2/a^2+y^2/b^2+z^2/c^2=0$, Imaginary cone.
6. $x^2/a^2+y^2/b^2-z^2/c^2=0$, Cone.
7. $x^2/a^2+y^2/b^2=2z/c$, Elliptic paraboloid.
8. $x^2/a^2-y^2/b^2=2z/c$, Hyperbolic paraboloid.
9. $x^2/a^2+y^2/b^2=1$, Elliptic cylinder.
10. $x^2/a^2-y^2/l^2=1$, Hyperbolic cylinder.
11. $x^2/a^2+y^2/b^2=-1$, Imaginary cylinder.
12. $x^2/a^2-y^2/b^2=0$, Pair of intersecting planes.
13. $x^2/a^2+y^2/b^2=0$, Pair of Imaginary planes.
14. $y^2=4ax$, Parabolic cylinder.
15. $y^2=a^2$, Two real parallel planes.
16. $y^2=-a^2$, Two imaginary planes.
17. $y^2=0$, Two coincident planes.

The equations representing cones and cylinders have already been considered and the reader is familiar with the nature of the surfaces represented by them.

In this chapter we propose to discuss the nature and some of the important geometrical properties of the surfaces represented by the equations 1, 2, 3, 4, 7, 8.

8·2. Shapes of some surfaces.

8·21. The Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

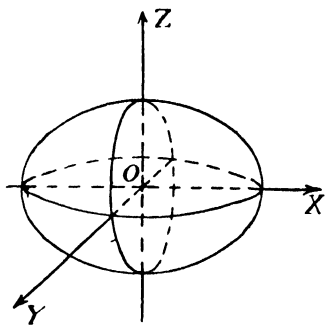

Fig. 26

The *following facts* enable us to trace the *locus* of this equation.

(*i*) If the co-ordinates x, y, z of *any* point satisfy the equation, then so do also the co-ordinates $-x, -y, -z$. But these points are on a straight line through the origin and are equidistant from the origin. Hence the origin bisects every chord which passes through it and is, on this account, called the *centre* of the surface.

(*ii*) If the point with co-ordinates x, y, z lies on the surface, then so does also the point $x, y, -z$. But the line joining these points is bisected at right angles by the XOY plane. Hence the XOY plane bisects every chord perpendicular to it and the surface is symmetrical with respect to this plane.

Similarly, the surface is symmetrical with respect to the YOZ and the ZOX planes.

These three planes are called **Principal Planes** in as much as they bisect all chords perpendicular to them. The three lines of intersection of the three principal planes taken in pairs are called **Principal axes.** Co-ordinate axes are the principal axes in the present case.

(*iii*) x cannot take a value which is numerically greater than a, for otherwise y^2 or z^2 would be negative. Similarly y and z cannot be numerically greater than b and c respectively.

Hence the surface lies between the planes

$$x=a,\ x=-a\ ;\ y=b,\ y=-b\ ;\ z=c,\ z=-c$$

and so is a *closed* surface.

(*iv*) The X-axis meets the surface in the two points $(a, 0, 0)$ and $(-a, 0, 0)$. Thus the surface intercepts a length $2a$ on X-axis. Similarly the lengths intercepted on Y and Z-axes are $2b$ and $2c$ respectively. Lengths $2a$, $2b$, $2c$ intercepted on the principal axes are called the lengths of the axes of the ellipsoid.

(*v*) The sections of the surface by the planes $z=k$ which are parallel to the XOY plane are similar ellipses having equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\ z=k\ ; \qquad \ldots(1)$$

k lying between $-c$ and c. These ellipses have their centres on Z-axis and diminish in size as k varies from 0 to c. The ellipsoid may, therefore, be generated by the variable ellipse (1) as k varies from $-c$ to c.

It may similarly be shown that the sections by planes parallel to the other co-ordinate planes are also ellipses and the ellipsoid may be supposed to be generated by them.

Note. The surface represented by the equation

$$x^2/a^2+y^2/b^2+z^2/c^2=-1.$$

which is not satisfied by any real values of x, y, z is imaginary.

8·22. The hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

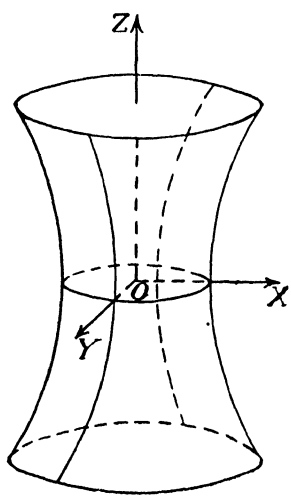

Fig. 27

(*i*) The origin bisects all chords through it and is, therefore, the centre of the surface.

(*ii*) The co-ordinate planes bisect all chords perpendicular to them and are, therefore, the planes of symmetry or the **Principal Planes** of the surface. The co-ordinate axes are its **Principal axes.**

(*iii*) The X-axis meets the surface in points $(a, 0, 0)$, $(-a, 0, 0)$ and thus the surface intercepts length $2a$ on X-axis. Similarly the length intercepted on Y-axis is $2b$, whereas Z-axis does not meet the surface in real points.

(*iv*) The sections by planes $z=k$ which are parallel to the XOY plane are the similar ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2},\ z=k \qquad \ldots(1)$$

whose centres lie on Z-axis and which increase in size as k increases. There is no limit to the increase of k. The surface may, therefore, be generated by the variable ellipse (1) where k varies from $-\infty$ to $+\infty$.

Again, sections by the planes $x=k$ and $y=k$ are hyperbolas

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{k^2}{a^2},\ x=k\ ;\quad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{k^2}{b^2},\ y=k$$

respectively.

Ex. Trace the surfaces

$$(i)\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ (ii)\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

8·23. The hyperboloid of two sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

(*i*) Origin is the **centre** ; co-ordinate planes are the **principal planes** ; and co-ordinate axes the **principal axes** of the surface.

(*ii*) X-axis meets the surface in the points $(a, 0, 0)$ and $(-a, 0, 0)$ whereas the Y and Z-axes meet the surface in imaginary points.

(*iii*) The sections by the planes $z=k$ and $y=k$ are the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{k^2}{c^2},\ z=k\ ;\quad \frac{x^2}{a^2}-\frac{z^2}{c^2}=1+\frac{k^2}{b^2},\ y=k$$

respectively.

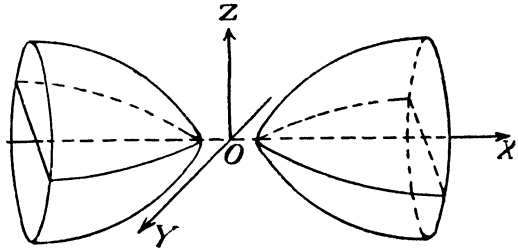

Fig. 28

The plane $x=k$ cuts the surface in the ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1,\ x=k,$$

which is imaginary for $-a<k<a$. Thus there is no portion of the surface included between the planes $x=-a$, $x=a$. When $k^2>a^2$ the section is a real ellipse which increases in size as k^2 increases.

The surface, therefore, consists of two detached portions.

Ex. Trace the surfaces

$$(i) \quad -\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad (ii) \quad -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

8·24. Central Conicoids. The four equations considered above are all included in the form

$$ax^2+by^2+cz^2=1. \qquad \ldots(1)$$

The surface is an ellipsoid if a, b, c are all positive, virtual ellipsoid, if all are negative, hyperboloid of one sheet, if two are positive and one negative ; and finally hyperboloid of two sheets if two are negative and one positive.

All *these surfaces have a centre and three principal planes* and are, therefore, known as *central conicoids.*

On the basis of the preceding discussion, the reader would do well to give precise definitions of (*i*) *Centre,* (*ii*) *Principal plane* and (*ii*) *Principal axis* of a conicoid.

In what follows, we shall consider the equation (1) and the geometrical results deducible from it will, therefore, hold in the case of all the central conicoids.

Ex. Show that the surface represented by the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=d$$

is a central conicoid ; origin being the centre.

Note. Cone is also a central conicoid, vertex being the centre ; this fact is clear from the general equation of a cone with its vertex at the origin.

8·3. Intersection of a line with a conicoid. *To find the points of intersection of the line*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

with the central conicoid

$$ax^2+by^2+cz^2=1. \qquad \ldots(ii)$$

Any point

$$(lr+\alpha,\ mr+\beta,\ nr+\gamma)$$

on the line (*i*) shall also lie on the surface (*ii*), if

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1,$$

or $$r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0. \qquad \ldots(A)$$

Let r_1, r_2 be the two roots of (*A*). Then

$$(lr_1+\alpha,\ mr_1+\beta,\ nr_1+\gamma),\ (lr_2+\alpha,\ mr_2+\beta,\ nr_2+\gamma)$$

are the two points of intersection.

Hence every line meets a central conicoid in two points.

We also see that *any plane section of central conicoid is a conic* for every line in the plane meets the curve of intersection in two points only.

The two values r_1 and r_2 of r obtained from equation (*A*) are the measures of the distances of the points of intersection P and Q from he point (α, β, γ) provided l, m, n are the *actual* direction cosines of he line.

Note. The equation (A) of this article will frequently be used in what follows.

Ex. 1. Find the points of intersection of the line

$$-\tfrac{1}{3}(x+5)=(y-4)=\tfrac{1}{7}(z-11)$$

with the conicoid

$$12x^2-17y^2+7z^2=7.$$

[*Ans.* (1, 2, —3), (—2, 3, 4).

2. Prove that the sum of the squares of the reciprocals of any three mutually perpendicular semi-diameters of a central conicoid is constant.

3. Any three mutually orthogonal lines drawn through a fixed point C meet the quadric

$$ax^2+by^2+cz^2=1$$

in P_1, P_2 ; Q_1, Q_2 ; R_1, R_2, respectively ; prove that

$$\frac{P_1P_2^2}{CP_1^2.CP_2^2}+\frac{Q_1Q_2^2}{CQ_1^2.CQ_2^2}+\frac{R_1R_2^2}{CR_1^2.CR_2^2}$$

and

$$\frac{1}{CP_1.CP_2}+\frac{1}{CQ_1.CQ_2}+\frac{1}{CR_1.CR_2}$$

are constants.

8·31. Tangent lines and tangent plane at a point.

Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

be *any* line through the point (α, β, γ) of the surface

$$ax^2+by^2+cz^2=1, \qquad \ldots(ii)$$

so that

$$a\alpha^2+b\beta^2+c\gamma^2=1. \qquad \ldots(iii)$$

One root of the equation (*A*) § 8·3 is, therefore, zero.

The line (*i*) will touch the conicoid (*ii*) at (α, β, γ) if both the values of r given by the equation (*A*) § 8·3 are zero.

The second value will also be zero, if

$$al\alpha+bm\beta+cn\gamma=0, \qquad \ldots(iv)$$

which is thus the condition for the line (*i*) to be a tangent line to the surface (*ii*) at (α, β, γ).

The locus of the tangent lines to the surface, at (α, β, γ), obtained by eliminating l, m, n between (*i*) and (*ii*), is

$$a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)=0,$$

or

$$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2=1,$$

which is a plane.

Hence the tangent lines at (α, β, γ) lie in the plane

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is, therefore, the *tangent plane* at (α, β, γ) to the conicoid

$$ax^2+by^2+cz^2=1.$$

Note. A tangent line at any point is a line which meets the surface in two coincident points and the tangent plane at a point is the locus of tangent lines at the point.

8·32. Condition of Tangency. *To find the condition that the plane*

$$lx+my+nz=p, \qquad \ldots(i)$$

should touch the conicoid

$$ax^2+by^2+cz^2=1. \qquad \ldots(ii)$$

If (α, β, γ) be the point of contact, the tangent plane

$$a\alpha x+b\beta y+c\gamma z=1, \qquad \ldots(iii)$$

thereat should be the same as the plane (i).

Comparing the two equations (i) and (iii), we get

$$\alpha=\frac{l}{ap},\ \beta=\frac{m}{bp},\ \gamma=\frac{n}{cp},$$

and since

$$a\alpha^2+b\beta^2+c\gamma^2=1,$$

we obtain the required condition

$$\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+\frac{\mathbf{n}^2}{\mathbf{c}}=\mathbf{p}^2.$$

Also the point of contact, then, is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right).$$

Thus we deduce that the planes

$$\mathbf{lx+my+nz=\pm\sqrt{(l^2/a+m^2/b+n^2/c)}}$$

touch the conicoid (ii) for all values of l, m, n.

8·33. Director Sphere. *To find the locus of the point of intersection of three mutually perpendicular tangent planes.*

Let

$$l_1x+m_1y+n_1z=\sqrt{\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)}, \qquad \ldots(i)$$

$$l_2x+m_2y+n_2z=\sqrt{\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)}, \qquad \ldots(ii)$$

$$l_3x+m_3y+n_3z=\sqrt{\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)} \qquad \ldots(iii)$$

be three mutually perpendicular tangent planes so that

$$\Sigma l_1m_1=\Sigma m_1n_1=\Sigma n_1l_1=0,$$
$$\Sigma l_1^2=\Sigma m_1^2=\Sigma n_1^2=1. \qquad \ldots(iv)$$

The co-ordinates of the point of intersection satisfy the three equations and its locus is, therefore, obtained by the elimination of l_1, m_1, n_1 ; l_2, m_2, n_2; l_3, m_3, n_3.

This is easily done by squaring and adding the three equations and using the relations (iv), so that we obtain

$$\mathbf{x^2+y^2+z^2=1/a+1/b+1/c,}$$

as the required locus which is a concentric sphere called the *Director sphere* of the given quadric.

Examples

1. *Find the equations to the tangent planes to*

$$7x^2-3y^2-z^2+21=0,$$

which pass through the line,

$$7x-6y+9=3,\ z=3.$$

Any plane

$$7x-6y+9+k(z-3)=0,$$

i.e.,

$$7x-6y+kz=3k-9,$$

through the given line will touch the given surface

$$7x^2-3y^2-z^2+21=0$$

i.e.,

$$-\tfrac{1}{3}x^2+\tfrac{1}{7}y^2+\tfrac{1}{21}z^2=1,$$

if

$$\frac{7^2}{-\frac{1}{3}}+\frac{(-6)^2}{\frac{1}{7}}+\frac{k^2}{\frac{1}{21}}=(3k-9)^2$$

i.e., if

$$2k^2+9k+4=0.$$

This gives

$$k=-4,\ -\tfrac{1}{2}.$$

Therefore the required planes are

$$7x-6y-4z+21=0,$$

$$7x-6y-\tfrac{1}{2}z+\tfrac{21}{2}=0.$$

2. *Obtain the tangent planes to the ellipsoid*

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

which are parallel to the plane

$$lx+my+nz=0.$$

If 2r is the distance between two parallel tangent planes to the ellipsoid, prove that the line through the origin and perpendicular to the planes lies on the cone

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0.$$

(D.U. Hons., 1947, 1959)

The tangent planes parallel to the plane

$$\Sigma lx=0,$$

are

$$\Sigma lx=\pm\sqrt{\Sigma a^2l^2}. \qquad \ldots(1)$$

The distance between these parallel planes which is twice the distance of either from the origin is

$$2\sqrt{\Sigma a^2l^2}/\sqrt{\Sigma l^2}.$$

Thus we have

$$\frac{2\sqrt{\Sigma a^2l^2}}{\sqrt{\Sigma l^2}}=2r,$$

or

$$\Sigma(a^2-r^2)l^2=0.$$

$\therefore$ the locus of the line

$$x/l=y/m=z/n,$$

which is perpendicular to the plane (1), is

$$\Sigma(a^2-r^2)x^2=0.$$

3. *The tangent planes to an ellipsoid at the points P_1, P_2, P_3, P_4 form a tetrahedron $A_1A_2A_3A_4$ where A_1 is the vertex which is not on the tangent plane at P_1. Prove that the planes*

$$A_1A_2P_2,\ A_1A_3P_3,\ A_1A_4P_4$$

have a line in common.

The tangent planes at points

$$P_1(x_1, y_1, z_1),\ P_2(x_2, y_2, z_2),\ P_3(x_3, y_3, z_3),\ P_4(x_4, y_4, z_4)$$

to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

are

$$(i)\ \frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1,\quad (ii)\ \frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=1,$$

$$(iii)\ \frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}=1,\quad (iv)\ \frac{xx_4}{a^2}+\frac{yy_1}{b^2}+\frac{zz_4}{c^2}=1,$$

respectively. The point A_1 is the intersection of the planes

$$(ii),\ (iii),\ (iv)$$

and A_2 is the intersection of the planes

$$(i),\ (iii),\ (iv).$$

Thus the line A_1A_2 is the line of intersection of the planes (*iii*) and (*iv*). Also P_2 is (x_2, y_2, z_2)

We may now easily show that the equation of the plane $A_1A_2P_2$ is

$$\left(\Sigma\frac{xx_3}{a^2}-1\right)\left(\Sigma\frac{x_2x_1}{a^2}-1\right)=\left(\Sigma\frac{xx_4}{a^2}-1\right)\left(\Sigma\frac{x_2x_3}{a^2}-1\right).$$

Similarly the two planes $A_1A_3P_3$ and $A_1A_4P_4$ are

$$\left(\Sigma\frac{xx_2}{a^2}-1\right)\left(\Sigma\frac{x_3x_4}{a^2}-1\right)=\left(\Sigma\frac{xx_4}{a^2}-1\right)\left(\Sigma\frac{x_3x_2}{a^2}-1\right),$$

$$\left(\Sigma\frac{xx_2}{a^2}-1\right)\left(\Sigma\frac{x_4x_3}{a^2}-1\right)=\left(\Sigma\frac{xx_3}{a^2}-1\right)\left(\Sigma\frac{x_4x_2}{a^2}-1\right).$$

From these it follows that these three planes all pass through the line

$$\left(\Sigma\frac{xx_3}{a^2}-1\right)\left(\Sigma\frac{x_2x_4}{a^2}-1\right)=\left(\Sigma\frac{xx_4}{a^2}-1\right)\left(\Sigma\frac{x_2x_3}{a^2}-1\right)$$
$$=\left(\Sigma\frac{xx_2}{a^2}-1\right)\left(\Sigma\frac{x_3x_4}{a^2}-1\right).$$

Hence the result.

Exercises

1. Show that the tangent planes at the extremities of any diameter of a central conicoid are parallel.

2. Show that the plane $3x+12y-6z-17=0$ touches the conicoid $3x^2-6y^2+9z^2+17=0$, and find the point of contact.

[*Ans.* $(-1, 2, 2/3)$.

3. Find the equations to the tangent planes to the surface

$$4x^2-5y^2+7z^2+13=0,$$

parallel to the plane

$$4x+20y-21z=0.$$

Find their points of contact also.

[*Ans.* $4x+20y-21z\pm13=0$; $(\pm1, \mp4, \mp3)$.

4. Find the equations to the two planes which contain the line given by

$$7x+10y-30=0,\ 5y-3z=0$$

and touch the ellipsoid

$$7x^2+5y^2+3z^2=60. \qquad (A.U.\ 1930)$$

[*Ans.* $7x+5y+3z-30=0,\ 14x+5y+9z-60=0$.

5. P, Q are any two points on a central conicoid. Show that the plane through the centre and the line of intersection of the tangent planes at P, Q will bisect PQ. Also show that if the planes through the centre parallel to the tangent planes at P, Q cut the chord PQ in P', Q', then

$$PP'=QQ'.$$

6. Prove that the locus of the foot of the central perpendicular on varying tangent planes of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

is the surface

$$(x^2+y^2+z^2)^2=a^2x^2+b^2y^2+c^2z^2. \qquad (B.\ U.\ 1915)$$

7. Find the locus of the perpendiculars from the origin to the tangent planes to the surface

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

which cut off from its axes intercepts the sum of whose reciprocals is equal to a constant $1/k$.

[*Ans.* $a^2x^2+b^2y^2+c^2z^2=k^2(x+y+z)^2$.

8. Show that the lines through (α, β, γ) drawn perpendicular to the tangent planes to

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

which pass through it generate the cone

$$[\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)]^2=a^2(x-\alpha)^2+b^2(y-\beta)^2+c^2(z-\gamma)^2.$$

9. If P is the point on the ellipsoid $x^2+2y^2+\frac{1}{3}z^2=1$ such that the perpendicular from the origin on the tangent plane at P is of unit length, show that P lies on one or other of the planes $3y=\pm z$.

8·34. Normal.

Def. *The normal at any point of a quadric is the line through the point perpendicular to the tangent plane thereat.*

The equation of the tangent plane at (α, β, γ) to the surface

$$ax^2+by^2+cz^2=1, \qquad \ldots(i)$$

is

$$a\alpha x+b\beta y+c\gamma z=1. \qquad \ldots(ii)$$

The equations to the normal at (α, β, γ), therefore, are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma} \qquad \ldots(iii)$$

so that $a\alpha$, $b\beta$, $c\gamma$ are the direction ratios of the normal.

If p, is the length of the perpendicular from the origin to the tangent plane (ii), we have

$$\frac{1}{a^2\alpha^2+b^2\beta^2+c^2\gamma^2}=p^2,$$

or

$$(a\alpha p)^2+(b\beta p)^2+(c\gamma p)^2=1,$$

which shows that $a\alpha p$, $b\beta p$, $c\gamma p$ are the actual direction cosines of the normal at (α, β, γ).

8·35. Number of normals from a given point.

If the normal (*iii*) at a point (α, β, γ) passes through a *given* point (f, g, h), then,

$$\frac{f-\alpha}{a\alpha}=\frac{g-\beta}{b\beta}=\frac{h-\gamma}{c\gamma}=r, \text{ (say)}$$

$$\therefore \quad \alpha=\frac{f}{1+ar}, \beta=\frac{g}{1+br}, \gamma=\frac{h}{1+cr}. \qquad \ldots(iv)$$

Since (α, β, γ) lies on the conicoid (*i*), we have the relation

$$\frac{af^2}{(1+ar)^2}+\frac{bg^2}{(1+br)^2}+\frac{ch^2}{(1+cr)^2}=1, \qquad \ldots(v)$$

which, being an equation of the *sixth* degree, gives six values of r, to each of which there corresponds a point (α, β, γ), as obtained from (*iv*).

Therefore there are *six* points on a central quadric the normals at which pass through a given point, *i.e., through a given point, six normals, in general, can be drawn to a central quadric.*

8·36. Cubic curve through the feet of normals. *The feet of the six normals from a given point to a central quadric are the intersections of the quadric with a certain cubic curve.*

Consider the curve whose parametric equations are

$$x=\frac{f}{1+ar}, \quad y=\frac{g}{1+br}, \quad z=\frac{h}{1+cr}, \qquad \ldots(vi)$$

where r is the *parameter.*

The points (x, y, z) on this curve, arising from those values of r which are the roots of the equation (*v*) are the six feet of the normals from the point (f, g, h).

Again, the points of intersection of this curve with any plane

$$Ax+By+Cz+D=0,$$

are given by

$$\frac{Af}{1+ar}+\frac{Bg}{1+br}+\frac{Ch}{1+cr}+D=0,$$

which determines three values of r. Hence the curve (*vi*) cuts *any* plane in three points and is, as such, a *cubic* curve.

Therefore, the six feet of the normals from (f, g, h) are the intersections of the conicoid and the cubic curve (*vi*).

8·37. Quadric cone through six concurrent normals. *The six normals drawn from any point to a central quadric are the generators of a quadric cone.*

We first prove that the lines drawn from (f, g, h) to intersect the cubic curve (*vi*) generate a quadric cone.

If *any* line

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n} \qquad \ldots(vii)$$

through (f, g, h) intersects the cubic curve, we have

$$\frac{f}{1+ar}-f = \frac{g}{1+br}-g = \frac{h}{1+cr}-h$$
$$\frac{l}{m}\qquad\qquad\frac{m}{n}$$

or
$$\frac{af/l}{1+ar} = \frac{bg/m}{1+br} = \frac{ch/n}{1+cr},$$

whence eliminating r, we get

$$\frac{af}{l}(b-c)+\frac{bg}{m}(c-a)+\frac{ch}{n}(a-b)=0,$$

which is the condition for the line (*vii*) to intersect the cubic curve (*vi*).

Eliminating l, m, n between the equations of the line and this condition, we get

$$\frac{af(b-c)}{x-f}+\frac{bg(c-a)}{y-g}+\frac{ch(a-b)}{z-h}=0,$$

which represents a cone of the second degree generated by lines drawn from (f, g, h) to intersect the cubic curve.

As the six feet of the normals drawn from (f, g, h) to the quadric lie on the cubic curve, the normals are, in particular, the generators of this cone of the second degree.

Note. The importance of this result lies in the fact that while five given concurrent lines determine a unique quadric cone, the six normals through a point lie on a quadric cone, *i.e*, *the quadric cone through any of the five normals through a point also contains the six normals through the point.*

8·38. The general equation of the conicoid through the six feet of the normals.

The co-ordinates (α, β, γ) of the foot of any of the six normals from (f, g, h) satisfy the relations

$$\frac{\alpha-f}{a\alpha}=\frac{\beta-g}{b\beta}=\frac{\gamma-h}{c\gamma}.$$

Hence we see that the feet of the normals lie on the three cylinders

$$ax(y-g)=by(x-f) \quad \text{or} \quad (a-b)xy-agx+bfy=0,$$
$$by(z-h)=cz(y-g) \quad \text{or} \quad (b-c)yz-bhy+cgz=0,$$
$$cz(x-f)=ax(z-h) \quad \text{or} \quad (c-a)zx-cfz+ahx=0.$$

The six feet of the normals are the common points of the three cylinders and the conicoid

$$ax^2+by^2+cz^2=1.$$

The equation

$$ax^2+by^2+cz^2-1+k_1[xy(a-b)-agx+bfy]+$$
$$k_2[yz(b-c)-bhy+cgz]+k_3[zx(c-a)-cfz+ahx]=0$$

is satisfied by the *six* feet of the normals and contains *three* arbitrary constants k_1, k_2, k_3. Therefore it represents the general equation of the conicoid through them.

Examples

1. *The normal at any point P of a central conicoid meets the three principal planes at G_1, G_2, G_3 ; show that PG_1, PG_2, PG_3, are in a constant ratio.*

The equations of the normal at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha p} = \frac{y-\beta}{b\beta p} = \frac{z-\gamma}{c\gamma p}.$$

Now since $a\alpha p$, $b\beta p$, $c\gamma p$, are the actual direction cosines each of these fractions represents the distance between the points

$$(\alpha, \beta, \gamma) \text{ and } (x, y, z).$$

Thus the distance PG_1, of the point $P(\alpha, \beta, \gamma)$ from the point G_1 where the normal meets the co-ordinate plane $x=0$ is

$$-1/ap.$$

Similarly $PG_2 = -1/bp$, $PG_3 = -1/cp$.

$$\therefore \quad PG_1 : PG_2 : PG_3 :: a^{-1} : b^{-1} : c^{-1}.$$

2. *Show that the lines drawn from the origin parallel to the normals to*

$$ax^2+by^2+cz^2=1.$$

at its points of intersection with the planes

$$lx+my+nz=p,$$

generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

Let f, g, h be any point on the curve of intersection of

$$ax^2+by^2+cz^2=1,\ lx+my+nz=p. \qquad \ldots(1)$$

The normal to the quadric at (f, g, h) is

$$\frac{x-f}{af} = \frac{y-g}{bg} = \frac{z-h}{ch}.$$

The line through the origin parallel to this normal is

$$\frac{x}{af} = \frac{y}{bg} = \frac{z}{ch}.$$

Also (f, g, h) satisfies the two equations (1) so that we have

$$af^2+bg^2+ch^2=1,\ lf+mg+nh=p. \qquad \ldots(3)$$

The required locus is obtained by eliminating f, g, h between (2) and (3).

The equations (3) give

$$af^2+bg^2+ch^2=\left(\frac{lf+mg+nh}{p}\right)^2, \qquad \ldots(4)$$

which is a second degree homogeneous expression in f, g, h. From (2) and (4), we can easily obtain the required locus.

3. *Prove that two normals to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

lie in the plane

$$lx+my+nz=0,$$

and the line joining their feet has direction cosines proportional to

$$a^2(b^2-c^2)mn,\ b^2(c^2-a^2)nl,\ c^2(a^2-b^2)lm.$$

Also obtain the co-ordinates of these points. (*M.T.*)

Let (f, g, h) be any point on the ellipsoid. The normal at the point is

$$\frac{x-f}{f/a^2}=\frac{y-g}{g/b^2}=\frac{z-h}{h/c^2}.$$

This lies in the given plane. if

$$lf+mg+nh=0,$$

$$lf/a^2+mg/b^2+nh/c^2=0.$$

These give

$$\frac{f/a}{amn(b^2-c^2)}=\frac{g/b}{bnl(c^2-a^2)}=\frac{h/c}{clm(a^2-b^2)}=\pm\frac{\sqrt{\Sigma f^2/a^2}}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}$$

$$=\pm\frac{1}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}.$$

Therefore the required two points are

$$\left[\pm\frac{a^2mn(b^2-c^2)}{d},\ \pm\frac{b^2nl(c^2-a^2)}{d},\ \pm\frac{c^2lm(a^2-b^2)}{d}\right]$$

where $$d=\sqrt{\Sigma a^2m^2n^2(b^2-c^2)}.$$

The direction cosines of the line joining these points are proportional to

$$a^2mn(b^2-c^2),\ \text{etc.}$$

4. *Prove that for all values of* λ, *the normals to the conicoid*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

which pass through a given point (α, β, γ) *meet the plane* $z=0$ *in points on the conic*

$$(b^2-c^2)\beta x+(c^2-a^2)\alpha y+(a^2-b^2)xy=0,\ z=0.$$

It can be shown that the equation of the quadric cone containing the normals to

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

drawn from the point (α, β, γ) is

$$\Sigma\frac{1}{a^2+\lambda}\,\alpha\left(\frac{1}{b^2+\lambda}-\frac{1}{c^2+\lambda}\right)\frac{1}{x-\alpha}=0, \qquad (\S\ 8{\cdot}37)$$

i.e.,

$$\Sigma\frac{\alpha(c^2-b^2)}{x-a}=0.$$

Thus it meets the plane $z=0$, where

$$\frac{\alpha(c^2-b^2)}{x-\alpha}+\frac{\beta(a^2-c^2)}{y-\beta}-(b^2-a^2)=0,$$

or $$\alpha(y-\beta)(c^2-b^2)+\beta(x-\alpha)(a^2-c^2)-(x-\alpha)(y-\beta)(b^2-a^2)=0,$$

or $$(b^2-c^2)\beta x+(c^2-a^2)\alpha y+(a^2-b^2)xy=0.$$

Exercises

1. If a point G be taken on the normal at any point P of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ such that

$$3PG=PG_1+PG_2+PG_3$$

show that the locus of G is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2}+\frac{b^2y^2}{(2b^2-c^2-a^2)^2}+\frac{c^2z^2}{(2c^2-a^2-b^2)^2}=\frac{1}{9}.$$

2. If a length PQ be taken on the normal at any point P of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

such that $PQ=k^2/p$ where k is a constant and p is the length of the perpendicular from the origin to the tangent plane at P, the locus of Q is

$$\frac{a^2x^2}{(a^2+k^2)^2}+\frac{b^2y^2}{(b^2+k^2)^2}+\frac{c^2z^2}{(c^2+k^2)^2}=1.$$

3. Show that, in general, two normals to the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

lie in a given plane. Determine the co-ordinates of the two points on the ellipsoid the normals at which lie in the plane

$$by-cz=\tfrac{1}{2}(b^2-c^2).$$

[*Ans.* $(\pm\sqrt{\tfrac{1}{2}}a, \tfrac{1}{2}b, \tfrac{1}{2}c)$.

4. Show that the locus of points on a central quadric, the normals at which intersect a given diameter is the curve of intersection with a cone having the principal axes of the quadric as generators.

5. Show that the normals at the points (x_1, y_1, z_1), and (x_2, y_2, z_2) to

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

intersect, if

$$\frac{(b^2-c^2)x_1}{x_1-x_2}+\frac{(c^2-a^2)y_1}{y_1-y_2}+\frac{(a^2-b^2)z_1}{z_1-z_2}=0,$$

and that if (f, g, h) be their point of intersection,

$$a^2f\left(\frac{1}{x_1}-\frac{1}{x_2}\right)=b^2g\left(\frac{1}{y_1}-\frac{1}{y_2}\right)=c^2h\left(\frac{1}{z_1}-\frac{1}{z_2}\right).$$

Deduce that the points on the surface, normals at which intersect the normal at a given point, lie on a quadric cone having its vertex at the given point.

6. Prove that six normals drawn from any point to a central conicoid meet a principal plane in six points which lie on a rectangular hyperbola.

7. The normals at six points on $x^2/a^2+y^2/b^2+z^2/c^2=1$ meet in the point (f, g, h); show that the mean position of the six points is

$$\left[\frac{-f(b^2+c^2-2a^2)a^2}{3(a^2-b^2)(a^2-c^2)}, \frac{-g(c^2+a^2-2b^2)b^2}{3(b^2-c^2)(b^2-a^2)}, \frac{-h(a^2+b^2-2c^2)c^2}{3(c^2-a^2)(c^2-b^2)}\right].$$

8·4. Plane of contact. The tangent plane

$$axx'+byy'+czz'=1,$$

at (x', y', z') to the quadric $ax^2+by^2+cz^2=1$, passes through (α, β, γ), if

$$a\alpha x'+b\beta y'+c\gamma z'=1.$$

This shows that the points on the quadric the tangent planes at which pass through the point (α, β, γ) lie on the plane

$$a\alpha x+b\beta y+c\gamma z=1,$$

which is called the plane of contact for the point (α, β, γ).

8·5. The polar plane of a point. *If any secant APQ through a given point A meets a conicoid in P and Q and a point R be taken on this line such that points A and R divide the line PQ internally and externally in the same ratio, then the locus of R is a plane called the polar plane of A.*

It may be easily seen that if the points A and R divide PQ internally and externally in the same ratio, then the points P, Q divide AR also internally and externally in the same ratio.

Let A, be a point (α, β, γ) and let (x, y, z) be the co-ordinates of R.

The co-ordinates of the point which divides AR in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x+\alpha}{\lambda+1}, \frac{\lambda y+\beta}{\lambda+1}, \frac{\lambda z+\gamma}{\lambda+1}\right).$$

This will lie on the conicoid

$$ax^2+by^2+cz^2=1,$$

for values of λ which are the roots of the equation

$$a\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2+b\left(\frac{\lambda y+\beta}{\lambda+1}\right)^2+c\left(\frac{\lambda z+\gamma}{\lambda+1}\right)^2=1$$

i.e., $\lambda^2(ax^2+by^2+cz^2-1)+2\lambda(a\alpha x+b\beta y+c\gamma z-1)$
$$+(a\alpha^2+b\beta^2+c\gamma^2-1)=0. \quad \ldots(1)$$

The two roots λ_1, λ_2 of this equation are the ratios in which the points P, Q divide the line AR. Since P, Q divide AR internally and externally in the same ratio, we have

$$\lambda_1+\lambda_2=0,$$

so that, from (1),

$$a\alpha x+b\beta y+c\gamma z-1=0. \quad \ldots(2)$$

Now (2) is the relation between the co-ordinates (x, y, z) of the point R. Being of the first degree, the equation (2) represents a plane.

Thus the polar plane of the point (α, β, γ) with respect to the conicoid

$$ax^2+by^2+cz^2=1.$$

is

$$a\alpha x+b\beta y+c\gamma z=1.$$

Any point is called the pole of its polar plane.

Note. The reader acquainted with cross ratios and, in particular, harmonic cross ratios, would know that the fact that the points P, Q divide AR internally and externally in the same ratio is also expressed by the statement

$$(AR, PQ)=-1.$$

This is further equivalent to the relation,

$$\frac{2}{AR}=\frac{1}{AP}+\frac{1}{AQ}.$$

Cor. The polar plane of a point on a conicoid coincides with the tangent plane thereat and that of a point outside it coincides with the plane of contact for that point.

Ex. 1. Show that the point of intersection of the tangent planes at three points on a quadric is the plane of the plane formed by their points of contact.

Ex. 2. Find the pole of the plane $lx+my+nz=p$ with respect to the quadric $ax^2+by^2+cz^2=1$. [*Ans.* l/ap, m/bp, n/cp).

8·51. Conjugate points and conjugate planes.

It is easy to show, that if the polar plane of a point P passes through another point Q, then the polar plane of Q passes through P.

Two such points are called *Conjugate points.*

Also, it can be shown that if the pole of a plane α lies on another plane β, then the pole of β lies on α.

Two such planes are called *Conjugate planes.*

8·52. Polar lines.

Consider any line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

The polar plane of any point $(lr+\alpha, mr+\beta, nr+\gamma)$ on this lir is

$$a(lr+\alpha)x+b(mr+\beta)y+c(nr+\gamma)z=1,$$

or

$$a\alpha x+b\beta y+c\gamma z-1+r(alx+bmy+cnz)=0,$$

which clearly passes through the line of intersection of the planes

$$a\alpha x+b\beta y+c\gamma z-1=0,$$

and

$$alx+bmy+cnz=0$$

for all values of r.

Thus the polar planes of all the points on a line l pass through another line l'.

Now, as the polar planes of any arbitrary point P on l passes through every point of l', therefore the polar planes of any point on l' will pass through the point P on l and, as P is arbitrary, it passes through every point on l, *i.e.*, passes through l.

Hence if the polar plane of any point on a line l passes through the line l', then the polar plane of any point on l' passes through l.

Two such lines are said to be *polar lines* with respect to the conicoid.

To find the polar line of any given line, we have only to find the line of intersection of the polar planes of any two points on it.

8·53. Conjugate lines.

Let l, m, be any two lines and l', m', their polar lines. Let m' *intersect* l at a point P.

We shall now show that the line l' also intersects the line m.

As P lies on m' and also on l, its polar plane contains the polar lines m and l' of m' and l respectively *i.e.*, the lines m and l' are coplanar and hence they intersect.

Hence if a line l intersects the polar of a line m, then the line m intersects the polar of the line l.

Two such lines l and m are *Conjugate lines.*

Example

Find the locus of straight lines drawn through a fixed point (α, β, γ) *at right angles to their polars with respect to*

$$ax^2+by^2+cz^2=1. \qquad (P.U.\ 1937)$$

Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

be any line perpendicular to its polar line. Now the polar line of (1) is the intersection of the planes

$$a\alpha x+b\beta y+c\gamma z=1,$$
$$alx+bmy+cnz=0.$$

If λ, μ, ν be the direction ratios of this line, we have

$$a\alpha\lambda+b\beta\mu+c\gamma\nu=0,$$
$$al\lambda+bm\mu+cn\nu=0.$$

These give

$$\frac{a\lambda}{n\beta-m\gamma}=\frac{b\mu}{l\gamma-n\alpha}=\frac{c\nu}{m\alpha-\beta l}.$$

Because of perpendicularity, we have

$$l\lambda+m\mu+n\nu=0.$$

$$\therefore \quad \frac{l(n\beta-m\gamma)}{a}+\frac{m(l\gamma-n\alpha)}{b}+\frac{n(m\alpha-l\beta)}{c}=0,$$

or $$\alpha mn\left(\frac{1}{b}-\frac{1}{c}\right)+\beta nl\left(\frac{1}{c}-\frac{1}{a}\right)+\gamma lm\left(\frac{1}{a}-\frac{1}{b}\right)=0,$$

or $$\frac{\alpha}{l}\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{\beta}{m}\left(\frac{1}{c}-\frac{1}{a}\right)+\frac{\gamma}{n}\left(\frac{1}{a}-\frac{1}{b}\right)=0. \qquad \ldots(2)$$

Eliminating l, m, n between (1) and (2), we see that the required locus is

$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{\beta}{y-\beta}\left(\frac{1}{c}-\frac{1}{a}\right)+\frac{\gamma}{z-\gamma}\left(\frac{1}{a}-\frac{1}{b}\right)=0.$$

Exercises

1. Prove that the locus of the poles of the tangent planes of

$$ax^2+by^2+cz^2=1,$$

with respect to

$$\alpha x^2+\beta y^2+\gamma z^2=1,$$

is the conicoid

$$\frac{\alpha^2x^2}{a}+\frac{\beta^2y^2}{b}+\frac{\gamma^2z^2}{c}=1.$$

2. Show that the locus of the poles of the plane

$$lx+my+nz=p,$$

with respect to the system of conicoids

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1,$$

where λ is the parameter, is a straight line perpendicular to the given plane.

3. Show that the polar line of

$$(x-1)/2=(y-2)/3=(z-3)/4$$

with respect to the quadric

$$x^2-2y^2+3z^2-4=0,$$

is

$$(x+6)/3=(y-2)/3=(z-2)/1.$$

4. Find the locus of straight lines drawn through a fixed point (f, g, h) whose polar lines with respect to the quadrics

$$ax^2+by^2+cz^2=1 \text{ and } \alpha x^2+\beta y^2+\gamma z^2=1$$

are coplanar.

$$[Ans. \quad \Sigma\frac{(\alpha-a)(b\gamma-c\beta)f}{x-f}=0.$$

5. Show that any normal to the conicoid

$$\frac{x^2}{pa+q}+\frac{y^2}{pb+q}+\frac{z^2}{pc+q}=1$$

is perpendicular to its polar line with respect to the conicoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1. \qquad (B.\ U.\ 1920)$$

6. Find the conditions that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

should be (*i*) polar, (*ii*) conjugate with respect to the coincoid

$$ax^2+by^2+cz^2=1.$$

$$[Ans. \quad (i)\ \Sigma a\alpha\alpha'=1,\ \Sigma a\alpha' l=0,\ \Sigma a\alpha l'=0,\ \Sigma all'=0$$
$$(ii)\ (\Sigma a\alpha l')\ (\Sigma a\alpha' l)=(\Sigma all')(\Sigma a\alpha\alpha'-1).$$

8·61. The enveloping cone. Def. *The locus of tangent lines to a quadric through any point is called the enveloping cone.*

To find the enveloping cone of the conicoid

$$ax^2+by^2+cz^2=1,$$

with its vertex at (α, β, γ).

Any line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(i)$$

through (α, β, γ) will meet the surface in two coincident points if the equation (A) of §8·3 has equal roots, *i.e.*, if

$$(al\alpha+bm\beta+cn\gamma)^2=(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1) \qquad \ldots(ii)$$

Eliminating l, m, n between (i) and (ii), we obtain

$$[a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)]^2$$
$$=[a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2](a\alpha^2+b\beta^2+c\gamma^2-1),$$

which is the required equation of the **enveloping cone.**

If we write

$$S\equiv ax^2+by^2+cz^2-1,\ S_1\equiv a\alpha^2+b\beta^2+c\gamma^2-1,\ T_1\equiv a\alpha x+b\beta y+c\gamma z-1,$$

we see that the equation of the enveloping cone can briefly be written as

$$(T_1-S_1)^2=(S-2T_1+S_1)S_1,$$

or

$$SS_1=T_1^2,$$

i.e., $(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)=(a\alpha x+b\beta y+c\gamma z-1)^2.$

Note. Obviously the enveloping cone passes through the points common to the conicoid and the polar plane $a\alpha x+b\beta y+c\gamma z=1$ of the vertex (α, β, γ).

Thus the enveloping cone may be regarded as a cone whose vertex is the given point and guiding curve is the section of the conicoid by its polar plane.

Exercises

1. A point P moves so that the section of the enveloping cone of $x^2/a^2+y^2/b^2+z^2/c^2=1$ with P as vertex by the plane $z=0$ is a circle ; show that P lies on one of the conics

$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1, x=0\,;\quad \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1, y=0.$$

2. If the section of the enveloping cone of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

whose vertex is P by the plane $z=0$ is a rectangular hyperbola, show that the locus of P is

$$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1. \qquad (Agra,\ 1938)$$

3. Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2+by^2+cz^2=1$.

[*Ans.* $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$.

4. A pair of perpendicular tangent planes to the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

passes through the fixed point $(0, 0, k)$. Show that their line of intersection lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

(D. U. Hons. 1949)

[The required locus is the locus of the line of intersection of perpendicular tangent planes to the enveloping cone of the given ellipsoid with vertex at $(0, 0, k)$.]

8·62. Enveloping Cylinder. Def. *The locus of tangent lines to a quadric parallel to any given line is called enveloping cylinder.*

To find the enveloping cylinder of the conicoid

$$ax^2+by^2+cz^2=1,$$

with its generators parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Let (α, β, γ) be any point on the *enveloping* cylinder, so that the equations of the generator through it are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(i)$$

As in § 8·61, the line (i) will touch the conicoid, if,

$$(al\alpha+bm\beta+cn\gamma)^2=(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1).$$

Therefore the locus of (α, β, γ) is the surface

$$(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2,$$

which is the required equation of the **enveloping cylinder.**

Note. **Equation of Enveloping cylinder deduced from that of Enveloping cone. Use of elements at infinity.** Since all the lines parallel to the line

$$x/l=y/m=z/n$$

pass through the point $(l, m, n, 0)$ which is, in fact, the point at infinity on each member of this system of parallel lines, we see that the enveloping cylinder is the enveloping cone with vertex $(l, m, n, 0)$.

The homogeneous equation of the surface being

$$ax^2+by^2+cz^2-t^2=0,$$

the equation of the enveloping cylinder is

$$(ax^2+by^2+cz^2-t^2)(al^2+bm^2+cn^2-0)=(alx+bmy+cnz-t.0)^2 \qquad (SS_1=T^2)$$

so that in terms of ordinary cartesian co-ordinates, this equation is

$$(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2.$$

Note. Clearly the generators of the enveloping cylinder touch the quadric at points where it is met by the plane $alx+bmy+cnz=0$ which is known as the plane of contact.

Exercises

1. Show that the enveloping cylinders of the ellipsoid

$$ax^2+by^2+cz^2=1,$$

with generators perpendicular to Z-axis meet the plane $z=0$ in parabolas.

2. Enveloping cylinders of the quadric $ax^2+by^2+cz^2=1$ meet the plane $z=0$ in rectangular hyperbola ; show that the central perpendiculars to their planes of contact generate the cone

$$b^2cx^2+a^2cy^2+ab(a+b)z^2=0.$$

3. Prove that the enveloping cylinders of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

whose generators are parallel to the lines,

$$\frac{x}{a}=\frac{y}{\pm\sqrt{(a^2-b^2)}}=\frac{z}{c},$$

meet the plane $z=0$ in circles. *(P. U. 1937)*

8·71. Locus of chords bisected at a given point. Section with a given centre.

Let the given point be (α, β, γ).

If any chord

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

of the quadric $ax^2+by^2+cz^2=1$ is bisected at (α, β, γ), the two roots r_1 and r_2 of the equation (A) of § 8·3 are equal and opposite so that $r_1+r_2=0$, and therefore

$$al\alpha+bm\beta+cn\gamma=0. \qquad \ldots(2)$$

Therefore the required locus, obtained by eliminating l, m, n, between (1) and (2), is

$$a\alpha(x-\alpha)+b\beta(y-\beta)+c\gamma(z-\gamma)=0,$$

which is a plane and can briefly be written as

$$T_1=S_1.$$

The section of the quadric by this plane is a conic whose centre is (α, β, γ) ; for this point bisects all chords of the conic through it.

Cor. *The plane which cuts* $ax^2+by^2+cz^2=1$, *in a conic whose centre is* (α, β, γ) *is*

$$\Sigma a\alpha x=\Sigma a\alpha^2.$$

Example

Triads of tangent planes at right angles are drawn to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. *Show that the locus of the centre of section of the surface by the plane through their points of contact is*

$$x^2+y^2+z^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2(a^2+b^2+c^2).$$

Suppose that (α, β, γ) is the centre of section of the surface by a plane through the points of contact of a triad of mutually perpendicular tangent planes. The pole of this section must thus be a point of the director sphere

$$x^2+y^2+z^2=a^2+b^2+c^2.$$

The equation of the section is $T_1=S_1$ *i.e.*,

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}. \qquad \ldots(i)$$

If (f, g, h) be its pole, the equation (i) must be the same as

$$\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}=1. \qquad \ldots(ii)$$

Comparing (i) and (ii), we have

$$f=\frac{\alpha}{\Sigma(\alpha^2/a^2)},\quad g=\frac{\beta}{\Sigma(\alpha^2/a^2)},\quad h=\frac{\gamma}{\Sigma(\alpha^2/a^2)}.$$

Since

$$f^2+g^2+h^2=a^2+b^2+c^2,$$

we have

$$\alpha^2+\beta^2+\gamma^2=[(\Sigma\alpha^2/a^2)]^2(a^2+b^2+c^2).$$

Replacing α, β, γ by x, y, z respectively, we have the required result.

Exercises

1. Find the equation to the plane which cuts the surface

$$x^2-2y^2+3z^2=4$$

in a conic whose centre is at the point (5, 7, 6).

[*Ans.* $5x-14y+18z=35$.

2. Find the centres of the conics

(*i*) $4x+9y+4z=-15$, $2x^2-3y^2+4z^2=1$;

(*ii*) $2x-2y-5z+5=0$, $3x^2+2y^2-15z^2=4$.

[*Ans.* (*i*) (2, −3, 1) (*ii*) (−2, 3, −1).

3. Prove that the plane through the three extremities of the different axes of a central conicoid cuts it in a conic whose centre coincides with the centroid of the triangle formed by those extremities.

4. Show that the centre of the conic

$$lx+my+nz=p,\ ax^2+by^2+cz^2=1$$

is the point

$$\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right),$$

where $l^2+m^2+n^2=1$ and $p_0=\sqrt{\Sigma l^2/a}$.

5. A variable plane makes intercepts on the axes of a central conicoid whose sum is zero. Show that the locus of the centre of the section determined by it is a cone which has the axes of the conicoid as its generators.

6. Find the locus of the centres of sections which pass through a given point.

7. Show that the centres of sections of $ax^2+by^2+cz^2=1$ by planes which are at a constant distance, p, from the origin lie on the surface

$$(ax^2+by^2+cz^2)=p^2(a^2x^2+b^2y^2+c^2z^2).$$

8. Find the locus of centres of sections of $ax^2+by^2+cz^2=1$, which touch $\alpha x^2+\beta y^2+\gamma z^2=1$.

[*Ans.* $a^2\alpha^{-1}x^2+b^2\beta^{-1}y^2+c^2\gamma^{-1}z^2=(ax^2+by^2+cz^2)^2$.

8·72. Locus of midpoints of a system of parallel chords. Let l, m, n be proportional to the direction cosines of a given system of parallel chords and let (α, β, γ) be the midpoint of any one of them.

As the chord

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

of the quadric is bisected at (α, β, γ), we have, as in § 8·71,

$$al\alpha+bm\beta+cn\gamma=0.$$

Now l, m, n, being fixed, the locus of the midpoints (α, β, γ) of the parallel chords is the plane

$$alx+bmy+cnz=0,$$

which clearly passes through the centre of the quadric and is known as the *diametral plane conjugate to the direction* l, m, n.

Conversely any plane $Ax+By+Cz=0$ through the centre is the *diametral plane* conjugate to the direction l, m, n given by

$$\frac{al}{A}=\frac{bm}{B}=\frac{cn}{C}.$$

Thus every central plane is a diametral plane conjugate to some direction.

Note. If P be any point on the conicoid, then the plane bisecting chords parallel to OP is called the *diametral plane of* OP.

Note. **Another method. Use of elements at infinity.** We know that the mid-point of any line AB is the harmonic conjugate of the point at infinity on the line *w.r.* to A and B. Thus the *locus of the mid-points of a system of parallel chords is the polar plane of the point at infinity common to the chords of the system.*

We know that $(l, m, n, 0)$ is the point at infinity lying on a line whose direction ratios are l, m, n. Its polar plane *w.r.* to the conicoid,

$$ax^2+by^2+cz^2-w^2=0,$$

expressed in cartesian homogeneous co-ordinates, is

$$alx+bmy+cnz-w.0=0,$$

i.e.,

$$alx+bmy+cnz=0.$$

Exercises

1. $P(1, 3, 2)$ is a point on the conicoid,

$$x^2-2y^2+3z^2+5=0.$$

Find the locus of the mid-points of chords drawn parallel to OP.

[*Ans.* $x-6y+6z=0$.

2. Find the equation of the chord of the quadric $4x^2-5y^2+6z^2=7$ through $(2, 3, 4)$ which is bisected by the plane $2x-5y+3z=0$.

[*Ans.* $(x-2)=\frac{1}{2}(y-3)=(z-4)$.

8·8. Conjugate diameters and diametral planes.

In what follows, we shall confine our attention to the ellipsoid only.

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The equation of the diametral plane bisecting chords parallel to OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0.$$

Let $Q(x_2, y_2, z_2)$ be any point on the section of the ellipsoid by this plane so that we have

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0,$$

which is the condition that the diametral plane of OP should pass through Q and, by symmetry, it is also the condition that the diametral plane of OQ should pass through P.

Thus if the diametral plane of OP passes through Q, then the diametral plane of OQ also passes through P.

Let $R(x_3, y_3, z_3)$ be one of the two points where the line of intersection of the diametral planes of OP and OQ meets the conicoid.

Since R is on the diametral planes OP and OQ, the diametral plane

$$\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}=0,$$

of OR passes through P and Q.

Thus we obtain the following two sets of relations :—

$$\left.\begin{aligned}&\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,\\&\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1,\\&\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1.\end{aligned}\right\}(A),\qquad \left.\begin{aligned}&\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0,\\&\frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0,\\&\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0.\end{aligned}\right\}(B)$$

The three semi-diameters OP, OQ, OR, which are such that the plane containing any two is the diametral plane of the third are called **conjugate semi-diameters.**

The co-ordinates of the extremities of the conjugate semi-diameters are connected by the relations A and B above.

The three diametral planes POQ, QOR, ROP which are such that each is the diametral plane of the line of intersection of the other two are called **Conjugate planes.**

We shall now obtain two more sets of relations C, D, equivalent to the relations A, B.

By virtue of the relations (A), we see that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c},$$

can be considered as the direction cosines of some three straight lines and the relations (B) show that these three straight lines are also mutually perpendicular.

Hence as in § 5·2,

$$\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}; \frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b}; \frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}$$

are also the direction cosines of three mutually perpendicular straight lines. Therefore, we have

$$\left.\begin{aligned} x_1^2+x_2^2+x_3^2&=a^2,\\ y_1^2+y_2^2+y_3^2&=b^2,\\ z_1^2+z_2^2+z_3^2&=c^2. \end{aligned}\right\} (C), \quad \left.\begin{aligned} y_1z_1+y_2z_2+y_3z_3&=0,\\ z_1x_1+z_2x_2+z_3x_3&=0,\\ x_1y_1+x_2y_2+x_3y_3&=0. \end{aligned}\right\} (D)$$

Properties of Conjugate Semi-diameters

8·81. *The sum of the squares of three conjugate semi-diameters is constant.*

Adding the relations (C), we get

$$OP^2+OQ^2+OR^2=a^2+b^2+c^2,$$

which is constant.

8·82. *The volume of the parallelopiped formed by three conjugate semi-diameters as coterminous edges is constant.*

The results (B) give

$$\frac{x_1/a}{\dfrac{y_2z_3-y_3z_2}{bc}}=\frac{y_1/b}{\dfrac{z_2x_3-z_3x_2}{ca}}=\frac{z_1/c}{\dfrac{x_2y_3-x_3y_2}{ab}}$$

$$=\frac{\sqrt{(\Sigma x_1^2/a^2)}}{\sqrt{\Sigma\left(\dfrac{y_2z_3-y_3z_2}{bc}\right)^2}}=\pm 1,$$

for

$$\sqrt{\Sigma\left(\frac{y_2z_3-y_3z_2}{bc}\right)^2}$$

is the sine of the angle between two perpendicular lines whose direction cosines are

$$\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} \text{ and } \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}.$$

$$\therefore \frac{x_1}{a}=\pm\frac{y_2z_3-y_3z_2}{bc}, \frac{y_1}{b}=\pm\frac{z_2x_3-z_3x_2}{ca}, \frac{z_1}{c}=\pm\frac{x_2y_3-x_3y_2}{ab}.$$

Now the volume of the parallelopiped whose coterminous edges are OP, OQ, OR

$$=6\times\text{volume of the tetrahedron } OPQR$$

$$=\begin{vmatrix} 0, & 0, & 0, & 1\\ x_1, & y_1, & z_1, & 1\\ x_2, & y_2, & z_2, & 1\\ x_3, & y_3, & z_3, & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$$

$$= x_1(y_2z_3 - y_3z_2) + y_1(z_2x_3 - z_3x_2) + z_1(x_2y_3 - x_3y_2)$$

$$= \pm \frac{bcx_1^2}{a} \pm \frac{cay_1^2}{b} \pm \frac{abz_1^2}{c}$$

$$= \pm abc \Sigma \frac{x_1^2}{a^2} = \pm abc, \text{ which is a constant.}$$

The same result can also be proved in the following manner :

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \times \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \begin{vmatrix} \Sigma x_1^2, & \Sigma x_1y_1, & \Sigma x_1z_1 \\ \Sigma x_1y_1, & \Sigma y_1^2, & \Sigma y_1z_1 \\ \Sigma x_1z_1, & \Sigma y_1z_1, & \Sigma z_1^2 \end{vmatrix}$$

(By the rule of multiplication of determinants)

$= a^2b^2c^2$, from (C) and (D).

8·83. *The sum of the squares of the areas of the faces of the parallelopiped formed with any three conjugate semi-diameters as coterminous edges is constant.*

Let A_1, A_2, A_3, be the areas of the triangles OQR, ORP, OPQ, and let l_i, m_i, n_i, $(i=1, 2, 3)$ be the direction cosines of the normals to the planes respectively.

Now the projection of the triangle OQR on the YZ plane is a triangle with vertices $(0, 0, 0)$, $(0, y_2, z_2)$, $(0, y_3, z_3)$ whose area is $\frac{1}{2}(y_2z_3 - y_3z_2)$. Also this is A_1l_1.

$$\therefore \qquad A_1l_1 = \tfrac{1}{2}(y_2z_3 - y_3z_2) = \pm \frac{bcx_1}{2a}.$$

Similarly

$$A_1m_1 = \pm \frac{cay_1}{2b}, \quad A_1n_1 = \pm \frac{abz_1}{2c}.$$

Squaring, we have

$$A_1^2 = \frac{b^2c^2x_1^2}{4a^2} + \frac{c^2a^2y_1^2}{4b^2} + \frac{a^2b^2z_1^2}{4c^2}.$$

Similarly projecting the areas ORP and OPQ on the co-ordinate planes, we get

$$A_2^2 = \frac{b^2c^2x_2^2}{4a^2} + \frac{c^2a^2y_2^2}{4b^2} + \frac{a^2b^2z_2^2}{4c^2},$$

$$A_3^2 = \frac{b^2c^2x_3^2}{4a^2} + \frac{c^2a^2y_3^2}{4b^2} + \frac{a^2b^2z_3^2}{4c^2}.$$

Adding we get

$$A_1^2+A_2^2+A_3^2=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2),$$

which is a constant.

8·84. *The sum of the squares of the projections of three semi-conjugate diameters on any line or plane is constant.*

Let l, m, n, be the direction cosines of any given line so that the sum of the squares of the projections of OP, OQ, OR on this line is

$$=(lx_1+my_1+nz_1)^2+(lx_2+my_2+nz_2)^2+(lx_3+my_3+nz_3)^2$$
$$=l^2\Sigma x_1^2+m^2\Sigma y_1^2+n^2\Sigma z_1^2+2lm\Sigma x_1y_1+2mn\Sigma y_1z_1+2nl\Sigma z_1x_1$$
$$=a^2l^2+b^2m^2+c^2n^2,$$

which is a constant.

Again, let l, m, n be the direction cosines of the normal to any given plane so that the sum of the squares of the projection of OP, OQ, OR on this plane is

$$=OP^2-(lx_1+my_1+nz_1)^2+OQ^2-(lx_2+my_2+nz_2)^2$$
$$+OR^2-(lx_3+my_3+nz_3)^2$$
$$=a^2+b^2+c^2-a^2l^2-b^2m^2-c^2n^2$$
$$=a^2(m^2+n^2)+b^2(n^2+l^2)+c^2(l^2+m^2),$$

which is a constant.

Examples

1. *Show that the equation of the plane through the extremities*

$$(x_k, y_k, z_k),\ k=1, 2, 3,$$

of the conjugate semi-diameters of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

is

$$\frac{x(x_1+x_2+x_3)}{a^2}+\frac{y(y_1+y_2+y_3)}{b^2}+\frac{z(z_1+z_2+z_3)}{c^2}=1.$$

If any plane

$$lx+my+nz=p$$

passes through the three extremities, then

$$lx_1+my_1+nz_1=p,$$
$$lx_2+my_2+nz_2=p,$$
$$lx_3+my_3+nz_3=p.$$

Multiplying by x_1, x_2, x_3, respectively, we obtain

$$la^2=p\Sigma x_1.$$

Similarly

$$mb^2=p\Sigma y_1,$$

and

$$nc^2=p\Sigma z_1.$$

Hence the required equation.

2. *Find the locus of the equal conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let OP, OQ, OR be three equal conjugate semi-diameters. We have

$$OP^2+OQ^2+OR^2=a^2+b^2+c^2\,;\ OP^2=OQ^2=OR^2.$$

$$\therefore \qquad OP^2=\tfrac{1}{3}(a^2+b^2+c^2).$$

Let P be the point (x_1, y_1, z_1). We require the locus of the line

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}, \qquad \ldots(1)$$

where

$$x_1^2+y_1^2+z_1^2=\tfrac{1}{3}(a^2+b^2+c^2), \qquad \ldots(2)$$

and

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1. \qquad \ldots(3)$$

From (2) and (3), we obtain the homogeneous relation

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=\frac{3(x_1^2+y_1^2+z_1^2)}{a^2+b^2+c^2}. \qquad \ldots(4)$$

Eliminating x_1, y_1, z_1 from (1) and (4), we obtain the required locus, *viz.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2}.$$

3. *Show that if the cone*

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0,$$

has three of its generators along conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

then

$$Aa^2+Bb^2+Cc^2=0.$$

Let OP, OQ, OR, where P, Q, R are the extremities of conjugate semi-diameters, be generators of the given cone.

Let

$$(x_1, y_1, z_1),\ (x_2, y_2, z_2),\ (x_3, y_3, z_3)$$

be the co-ordinates of these points. Since these points lie on the given cone, we have

$$Ax_1^2+By_1^2+Cz_1^2+2Fy_1z_1+2Gz_1x_1+2Hx_1y_1=0,$$

and two similar results.

Adding these three results and making use of the relations C and D of § 8·8, we obtain the given relation.

4. *With any point on the surface of any ellipsoid as centre, a sphere is described such that the tangent planes can be drawn to it from the centre of the ellipsoid which are conjugate diametral planes of the ellipsoid. Show that its radius is the same for all positions of its centre.*

Consider any point (f, g, h) on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let the three conjugate diametral planes

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0,\quad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=0,\quad \frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}=0,$$

be tangent planes to a sphere with centre (f, g, h) and radius r. The distance of (f, g, h) from each of the three planes being equal to r, we have

$$r=\frac{\Sigma(fx_1/a^2)}{\sqrt{(\Sigma x_1^2/a^4)}} \quad \text{or} \quad r^2\Sigma\frac{x_1^2}{a^4}=\left(\Sigma\frac{fx_1}{a^2}\right)^2,$$

$$r=\frac{\Sigma(fx_2/a^2)}{\sqrt{(\Sigma x_2^2/a^4)}} \quad \text{or} \quad r^2\Sigma\frac{x_2^2}{a^4}=\left(\Sigma\frac{fx_2}{a^2}\right)^2,$$

$$r=\frac{\Sigma(fx_3/a^2)}{\sqrt{(\Sigma x_3^2/a^4)}} \quad \text{or} \quad r^2\Sigma\frac{x_3^2}{a^4}=\left(\Sigma\frac{fx_3}{a^2}\right)^2.$$

Adding and making use of the relations C and D of § 8·8, we have

$$r^2\left[\Sigma\frac{1}{a^2}\right]=\Sigma\frac{f^2}{a^2}=1$$

or
$$r^2=(\Sigma a^{-2})^{-1}.$$

Hence the result.

Exercises

1. Show that the lines

$$\frac{x}{1}=\frac{y}{4}=\frac{z}{3},\quad \frac{x}{4}=\frac{y}{1}=\frac{z}{-9},\quad \frac{x}{26}=\frac{y}{-28}=\frac{z}{45},$$

are three mutually conjugate diameters of the ellipsoid

$$\frac{x^2}{2}+\frac{y^2}{4}+\frac{z^2}{9}=1.$$

2. Find the equations of the diameter in the plane $x+y+z=0$, conjugate to $x=-\frac{1}{2}y=\frac{1}{3}z$ with respect to the conicoid $3x^2+y^2-2z^2=1$. What are the equations of the third conjugate diameter ?

$$\left[Ans.\ \frac{x}{4}=\frac{y}{-9}=\frac{z}{5};\quad \frac{x}{34}=\frac{y}{42}=\frac{z}{3}.\right.$$

3. Show that for the ellipsoid $x^2+4y^2+5z^2=1$, the two diameters $\frac{1}{3}x=-\frac{1}{2}y=\frac{1}{4}z$ and $x=0$, $2y=5z$ are conjugate. Obtain the equation of the third conjugate diameter. *Ans.* $x/16=y=-z/2$.

4. If p_1, p_2, p_3 ; π_1, π_2, π_3, be the projections of three conjugate diameters on any two given lines, then $p_1\pi_1+p_2\pi_2+p_3\pi_3$ is constant.

5. If three conjugate diameters vary so that OP, OQ lie respectively in the fixed planes

$$\frac{\alpha_1 x}{a^2}+\frac{\beta_1 y}{b^2}+\frac{\gamma_1 z}{c^2}=0,\quad \frac{\alpha_2 x}{a^2}+\frac{\beta_2 y}{b^2}+\frac{\gamma_2 z}{c^2}=0;$$

show that the locus of OR is the cone

$$\Sigma a^2(\beta_1 z-\gamma_1 y)(\beta_2 z-\gamma_2 y)=0.$$

[The required locus of OR is obtained from the fact that the lines of intersection of the diametral plane of OR with the given planes are conjugate lines.]

6. From a fixed point H perpendiculars HA, HB, HC are drawn to the conjugate diameters OP, OQ, OR respectively ; show that

$$OP^2.HA^2+OQ^2.HB^2+OR^2.HC^2$$

is constant.

7. OP, OQ, OR are conjugate diameters of an ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1.$$

At Q and R tangent lines are drawn parallel to OP and p_1, p_2 are their distances from O. The perpendicular from O to the tangent plane at right angles to OP is p.

Prove that

$$p^2+p_1{}^2+p_2{}^2=a^2+b^2+c^2.$$

(D.U. Hons. 1945)

8. Show that the plane $lx+my+nz=p$ will pass through the extremities of conjugate semi-diameters if

$$a^2l^2+b^2m^2+c^2n^2=3p^2.$$

9. Show that the locus of the centre of the section of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

by the plane PQR is the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1/3.$$

Prove that this locus coincides with the locus of the centroid of the triangle PQR.

10. Prove that the plane PQR touches the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1/3,$$

at the centroid of the triangle PQR. *(D.U. Hons. 1948)*

11. Find the locus of the foot of the perpendicular from the centre of the ellipsoid to the plane PQR.

[*Ans.* $a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2$.

12. If one of the three extremities $P(x_1, y_1, z_1)$ of conjugate diameters be kept fixed, show that the locus of the line joining the centre to the centroid of the triangle PQR is the cone

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}\right)^2.$$

13. If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the extremities of three conjugate diameters of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

show that the equation of the plane through the three points

$$(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3)$$

is

$$\left(\frac{x_1}{a^2}+\frac{y_1}{b^2}+\frac{z_1}{c^2}\right)x+\left(\frac{x_2}{a^2}+\frac{y_2}{b^2}+\frac{z_2}{c^2}\right)y+\left(\frac{x_3}{a^2}+\frac{y_3}{b^2}+\frac{z_3}{c^2}\right)z=1,$$

and that it touches the sphere

$$(x^2+y^2+z^2)\,(a^{-2}+b^{-2}+c^{-2})=1.$$

14. The enveloping cone from a point P to the ellipsoid $\sum x^2/a^2=1$ has three generating lines parallel to conjugate diameters of the ellipsoid ; show that the locus of P is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3}{2}x.$$

(B.U. 1958)

15. Show that any two sets of conjugate diameters of the ellipsoid lie on a quadric cone. (Deduce from Example 3, page 171).

Paraboloids

8·9. Having discussed the nature and geometrical properties of central conicoids, we now proceed to the consideration of *paraboloids.*

8·91. The Elliptic Paraboloids $x^2/a^2+y^2/b^2=2z/c$.

We have the following particulars about this surface :

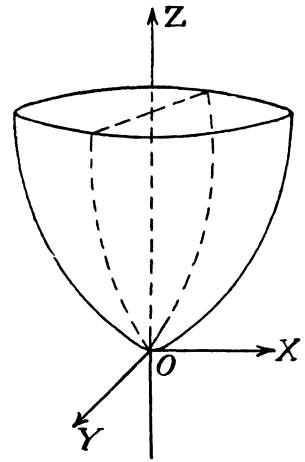

Fig. 29

(*i*) The co-ordinate planes $x=0$ and $y=0$ bisect chords perpendicular to them and are, therefore, its two planes of symmetry or **Principal planes.**

(*ii*) z cannot be negative, and hence there is no part of the surface on the negative side of the plane $z=0$. We have taken c positive.

(*iii*) The sections by the planes $z=k$, $(k>0)$, parallel to the XY plane, are similar ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c},\ z=k \qquad \ldots(i)$$

whose centres lie on Z-axis and which increase in size as k increases; there being no limit to the increase of k. The surface may thus be supposed to be generated by the variable ellipse (*i*).

Hence the surface is entirely on the positive side of the plane $z=0$, and extends to infinity.

(*iv*) The section of the surface by planes parallel to the YZ and ZX planes are clearly parabolas.

The Fig. 29 shows the nature of the surface.

Ex. Trace the surface $x^2/a^2+y^2/b^2=-2z/c$. $(c>0)$

8·92. The Hyperbolic Paraboloid $x^2/a^2-y^2/b^2=2z/c$.

(*i*) The co-ordinate planes $x=0$, $y=0$ are the two **Principal planes.**

(*ii*) The sections by the planes $z=k$ are the similar hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c},\ z=k,$$

with their centres on Z-axis.

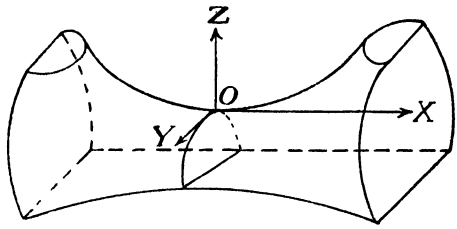

Fig. 30.

If k be positive, the real axis of the hyperbola is parallel to X-axis, and if k be negative, the real axis is parallel to Y-axis.

The section by the plane $z=0$ is the pair of lines

$$\frac{x}{a}=\frac{y}{b},\ z=0 \text{ and } \frac{x}{a}=-\frac{y}{b},\ z=0.$$

(*iii*) The section by the planes parallel to YZ and ZX planes are parabolas.

The Fig. 30 shows the nature of the surface.

Note. The two equations considered in the last two articles are clearly both included in the form

$$ax^2+by^2=2cz.$$

This equation represents an elliptic paraboloid if a and b are both positive or both negative, and a hyperbolic paraboloid if one is positive and the other negative.

Hence for an elliptic paraboloid ab is positive but, for hyperbolic paraboloid, ab is negative.

The geometrical results deducible from the equation $ax^2+by^2=2cz$ will hold for both the types of paraboloids.

Note. The reader would do well to give precise definitions of (*i*) vertex, (*ii*) principal planes, (*iii*) axis of a paraboloid.

8·93. Intersection of a line with a paraboloid.

The points of intersection of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

with the paraboloid

$$ax^2+by^2=2cz,$$

are

$$(lr+\alpha,\ mr+\beta,\ nr+\gamma)$$

for the *two* values of r which are the roots of the quadric equation

$$r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \quad \ldots(A)$$

We thus see that every line meets a paraboloid in two points.

It follows from this that the *plane sections of paraboloids are conics.*

Also, if $l=m=0$, one value of r is infinite and hence any line parallel to Z-axis meets the paraboloid in one point at an infinite distance from (α, β, γ) and so meets it in one finite point only. Such lines are called **diameters of the paraboloid.**

In particular, Z-axis meets the surface at the origin only.

8·94. From the equation (A) § 8·93 above, we deduce certain results similar to those obtained for central conicoids. The proofs of some of them are left as an exercise to the student.

1. The *tangent plane to* $ax^2+by^2=2cz$ at any point (α, β, γ) on the surface is

$$a\alpha x+b\beta y=c(z+\gamma).$$

In particular, $z=0$ is the tangent plane at the origin and Z-axis is the normal thereat.

The origin O is called the **vertex** of the paraboloid and Z-axis, the **axis** of the paraboloid.

2. Condition of Tangency. *The condition that the plane*

$$lx+my+nz=p,$$

may touch the paraboloid

$$ax^2+by^2=2cz, \qquad \ldots(1)$$

is

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{2np}{c}=0,$$

and the point of contact, then, is

$$\left(\frac{-lc}{an}, \frac{-mc}{bn}, \frac{-p}{n}\right).$$

Thus the plane

$$2n(lx+my+nz)+c(l^2/a+m^2/b)=0,$$

touches the surface (1) *for all values of* l, m, n.

3. Locus of the point of intersection of the three mutually perpendicular tangent planes.

If

$$2n_r(l_rx+m_ry+n_rz)+c\left(\frac{l_r^2}{a}+\frac{m_r^2}{b}\right)=0, \quad (r=1, 2, 3)$$

be three mutually perpendicular tangent planes, the locus of their point of intersection is obtained by eliminating l_r, m_r, n_r, which is done by adding the three equations and is, therefore,

$$\mathbf{2z+c\left(\frac{1}{a}+\frac{1}{b}\right)=0,}$$

and is a plane at right angles to the Z-axis ; the axis of the paraboloid.

4. Equations of the *normal* at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}.$$

5. The *polar plane* of the point (α, β, γ) is

$$a\alpha x+b\beta y=c(\gamma+z).$$

6. The equation of the *enveloping cone* with the point (α, β, γ) as its vertex is $SS_1 = T_1^2$, *i.e.*,

$$(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma)=(a\alpha x+b\beta y-cz-c\gamma)^2.$$

Its *plane of contact* with the paraboloid is the polar plane

$$a\alpha x+b\beta y-cz-c\gamma=0$$

of the vertex (α, β, γ).

7. The equation of the *enveloping cylinder* having its generators parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

is

$$(ax^2+by^2-2cz)(al^2+bm^2)=(alx+bmy-cn)^2.$$

Its *plane of contact* is the plane

$$alx+bmy-cn=0.$$

8. The *locus of chords bisected at a point* (α, β, γ) is the plane $T_1=S_1$, *i.e.*,

$$a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)=0.$$

This plane will meet the paraboloid in a conic whose centre is at (α, β, γ).

9. The *locus of mid-point of a system of parallel chords* with direction ratios l, m, n, is the plane

$$alx+bmy-cn=0$$

which is parallel to Z-axis, the axis of the paraboloid. The plane is called a *diametral plane* conjugate to the given direction.

Any plane $Ax+By+D=0$ parallel to the axis of the paraboloid is easily seen, by comparison, to be the diametral plane for the system of parallel chords with direction ratios

$$A/a,\ B/b,\ -D/c.$$

Any plane parallel to the axis of a paraboloid is, thus, a diametral plane.

Exercises

1. Show that

(*i*) the plane $2x-4y-z+3=0$ touches the paraboloid

$$x^2-2y^2=3z\ ;$$

(*ii*) the plane $8x-6y-z=5$ touches the paraboloid

$$x^2/2-y^2/3=z\ ;$$

and find the co-ordinates of the points of contact. (*D.U. Hons. 1958*)

[*Ans.* (*i*) (3, 3, −3), (*ii*) (8, 9, 5).

2. Show that the equation to the two tangent planes to the surface

$$ax^2+by^2=2z,$$

which passes through the line

$$u\equiv lx+my+nz-p=0,\ u'\equiv l'x+m'y+n'y-p'=0,$$

is

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}-2n'p'\right)-2uu'\,(ll'+mm'-np'-n'p)+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}-2np\right)=0.$$

3. Tangent planes at two points P and Q of a paraboloid meet in the line RS; show that the plane through RS and the middle point of PQ is parallel to the axis of the paraboloid.

4. Find the equation of the plane which cuts the paraboloid

$$x^2-\tfrac{1}{2}y^2=z$$

in a conic with its centre at the point (2, 3, 4).

[*Ans.* $4x-3y-z+5=0$.

5. Show that the locus of the centres of a system of parallel plane sections of a paraboloid is a diameter.

6. Show that the centre of the conic

$$ax^2+by^2=2z,\ lx+my+nz=p,$$

is the point

$$\left(-\frac{l}{an}, -\frac{m}{bn}, \frac{k^2}{n^2}\right),$$

where

$$k^2=\frac{l^2}{a}+\frac{m^2}{b}+np.$$

7. Find the chord through the point (2, 3, 4) which is bisected by the diametral plane $10x-24y=21$ of the paraboloid $5x^2-6y^2=7z$.

[*Ans.* $(x-2)=\tfrac{1}{2}(y-3)=\tfrac{1}{3}(z-4)$.

8·95. Number of normals from a given point.

If the normal at (α, β, γ) passes through a given point (f, g, h), then

$$\frac{f-\alpha}{a\alpha}=\frac{g-\beta}{b\beta}=\frac{h-\gamma}{-c}=r, \text{ (say)}$$

so that

$$\alpha=\frac{f}{1+ar},\ \beta=\frac{g}{1+br},\ \gamma=h+cr. \qquad \ldots(i)$$

Since (α, β, γ) lies on the paraboloid, we have the relation

$$a\frac{f^2}{(1+ar)^2}+b\frac{g^2}{(1+br)^2}=2c(h+cr), \qquad \ldots(ii)$$

which, being an equation of the fifth degree in r, gives five values of r, to each of which there corresponds a point (α, β, γ), from (i).

Therefore there are *five* points on a paraboloid the normals at which pass through a given point, *i.e., through a given point five normals, in general, can be drawn to a paraboloid.*

Cor. 1. As in § 8·36, page, 195, it can be shown that the feet of the five normals from the point, (f, g, h) to the surface are the points of intersection of the surface with the cubic curve

$$x=\frac{f}{1+ar},\ y=\frac{g}{1+br},\ z=h+cr, \qquad \ldots(iii)$$

where r is the parameter.

Cor. 2. Lines drawn from (f, g, h) to intersect the cubic curve (iii) generate the quadric cone

$$\frac{f}{x-f}-\frac{g}{y-g}+\frac{c(b-a)}{ab(z-h)}=0,$$

and, in particular, this cone contains the five normals from (f, g, h) as its generators.

8·96. Conjugate diametral planes.

Consider any two *diametral planes*

$$lx+my+p=0, \qquad \ldots(i)$$

and

$$l'x+m'y+p'=0. \qquad \ldots(ii)$$

The plane (i) bisects chords parallel to the line

$$\frac{x}{l/a}=\frac{y}{m/b}=\frac{z}{-p/c}. \qquad \ldots(iii)$$

which will be parallel to the plane (ii), if

$$\frac{ll'}{a}+\frac{mm'}{b}=0. \qquad \ldots(iv)$$

The symmetry of the result shows that the plane (i) is also parallel to the chords bisected by the plane (ii).

Thus if α *and* β *be two diametral planes*, such that *the plane* α *is parallel to the chords bisected by the plane* β, *then* β *is parallel to the chords bisected by* α.

Two such planes are called *Conjugate diametral planes.*

Equation (iv) is the condition for the diametral planes (i) and (ii) to be conjugate.

Ex. Show that the diametral planes

$$x+3y+3,\ 2x-y=1$$

are conjugate for the paraboloid

$$2x^2+3y^2=4z.$$

CHAPTER IX

PLANE SECTIONS OF CONICOIDS

9·1. We have seen that all plane sections of a conicoid are conics. We now proceed to determine the nature, the lengths, and the direction ratios of the axes of any plane section of a given conicoid.

We shall first consider the sections of central conicoids, and then of paraboloids.

While determining the nature of plane sections of conicoids, we shall *assume* that the orthogonal projection of a parabola is another parabola, of a hyperbola another hyperbola and of an ellipse is another ellipse or in some cases a circle.

9·2. Nature of the plane section of a central conicoid. *To determine the nature of the section of the central conicoid.*

$$ax^2+by^2+cz^2=1, \qquad \ldots(1)$$

by the plane

$$lx+my+nz=p. \qquad \ldots(2)$$

The equation to the cylinder passing through the section and having its generators parallel to Z-axis, obtained by eliminating z from (1) and (2), is

$$x^2(an^2+cl^2)+2clmxy+y^2(bn^2+cm^2)-2cplx-2cpmy+(cp^2-n^2)=0.$$

The plane $z=0$ which is perpendicular to the generating lines of the cylinder cuts it in the conic whose equations are

$$z=0,$$

$$x^2(an^2+cl^2)+2clmxy+y^2(bn^2+cm^2)-2cplx-2cpmy+(cp^2-n^2)=0,$$

and which is the projection of the given section on the plane

$$z=0.$$

The projection and, therefore, also the given section is a parabola, hyperbola or ellipse according as

$$c^2l^2m^2 \begin{cases} = \\ > \\ < \end{cases} (an^2+cl^2)(bn^2+cm^2)$$

or $$bcl^2+cam^2+abn^2 \begin{cases} = \\ <0. \\ > \end{cases}$$

Thus we find that the section is

a parabola, *a hyperbola*, *an ellipse* } *according as* $bcl^2+cam^2+abn^2 \begin{cases} = \\ <0 \\ >. \end{cases}$

9·21. Axes of central plane section. *To determine the lengths and direction cosines of the section of the central conicoid*

$$ax^2+by^2+cz^2=1, \qquad \ldots(1)$$

by the central plane

$$lx+my+nz=0. \qquad \ldots(2)$$

Take a concentric sphere

$$x^2+y^2+z^2=r^2. \qquad \ldots(3)$$

The extremities of all the semi-diameters of length r of the conicoid lie on the curve of intersection of the conicoid and the sphere.

The lines joining the origin to the points on this curve form a cone whose equation, obtained by making (1) and (3) homogeneous, is

$$(ar^2-1)x^2+(br^2-1)y^2+(cr^2-1)z^2=0. \qquad \ldots(4)$$

The plane (2) cuts this cone in two generators which determine the directions of two equal diameters of length $2r$ of the section and which are, therefore, equally inclined to the axes of the section.

In case $2r$ becomes the length of either axis of the section, the generators coincide and, therefore, the plane touches the cone, the generator of contact being one of the axes.

Now, the condition for the plane (2) to touch the cone (4) is

$$\frac{l^2}{ar^2-1}+\frac{m^2}{br^2-1}+\frac{n^2}{cr^2-1}=0.$$

or
$$(bcl^2+cam^2+abn^2)r^4-[(b+c)l^2+(c+a)m^2+(a+b)n^2]r^2 +(l^2+m^2+n^2)=0, \qquad \ldots(5)$$

which is a quadric in r^2 and has two roots r_1^2, r_2^2 which are the squares of the semi-axes of the section.

If λ, μ, ν be the direction ratios of the axis of length $2r$, the plane (2) touches the cone (4) along the line

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}, \qquad \ldots(6)$$

and is, therefore, identical with

$$(ar^2-1)\lambda x+(br^2-1)\mu y+(cr^2-1)\nu z=0,$$

which is the equation of the tangent plane at any point of the line (6) so that we have

$$\frac{\lambda(ar^2-1)}{l}=\frac{\mu(br^2-1)}{m}=\frac{\nu(cr^2-1)}{n}$$

which determine the direction ratios of the axis of length $2r$; r being given by the equation (5).

9·22. Areas of plane sections.

If the plane section be an ellipse,

$$\text{its area}=\pi r_1 r_2=\pi\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(bcl^2+cam^2+abn^2)}}.$$

If, p, be the length of the perpendicular from the origin to the tangent plane to the conicoid,

$$lx+my+nz=\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)},$$

which is parallel to the given plane $lx+my+nz=0$, we have

$$p=\frac{\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}}{\sqrt{(l^2+m^2+n^2)}}=\frac{\sqrt{(bcl^2+cam^2+abn^2)}}{\sqrt{(l^2+m^2+n^2)}}\cdot\sqrt{\frac{1}{abc}}$$

so that

$$\text{the area}=\frac{\pi}{p\sqrt{(abc)}}.$$

9·23. Condition for the section to be a rectangular hyperbola. For a rectangular hyperbola, we have

$$r_1^2+r_2^2=0,$$

and hence

$$\mathbf{(b+c)l^2+(c+a)m^2+(a+b)n^2=0.}$$

Ex. Obtain the condition that the section of the conicoid

$$ax^2+by^2+cz^2=1$$

by the plane $lx+my+nz=p$ should be a parabola, an ellipse, a hyperbola or a circle from the equation 5 of § 9·21

(For a circle $r_1^2=r_2^2$)

[*Ans.* The conditions for a circle are

$l=0$, $m^2(c-a)=n^2(a-b)$; or $m=0$, $n^2(a-b)=l^2(b-c)$;

or $n=0$, $l^2(b-c)=m^2(c-a)$.

9·24. *To find the condition for two lines*

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1},\quad \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2} \qquad \ldots(1)$$

to be the axes of the section by the plane through the same.

The quadric is

$$ax^2+by^2+cz^2=1$$

As each of the two lines in (1) will bisect chords of the section parallel to the other, we see that each of them must belong to the diametral plane conjugate to the other.

Now the diametral plane conjugate to

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$$

is

$$al_1x+bm_1y+cn_1z=0,$$

and the condition for the same to contain the second line is

$$\mathbf{al_1l_2+bm_1m_2+cn_1n_2=0.} \qquad \ldots(2)$$

The condition (2) is the one sought.

In addition to (2), we also have

$$l_1l_2+m_1m_2+n_1n_2=0,$$

for the axes are necessarily perpendicular.

Examples

1. *Planes are drawn through the origin so as to cut the quadric*

$$ax^2+by^2+cz^2=1,$$

in rectangular hyperbolas. Prove that the normals to the planes through the origin lie on a quadric cone.

Consider any plane

$$lx+my+nz=0 \quad ..(1)$$

through the origin. The condition for this plane to cut the given quadric in a rectangular hyperbola is

$$(b+c)l^2+(c+a)m^2+(a+b)n^2=0. \quad ...(2)$$

The normal to the plane (1) through the origin is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \quad ...(3)$$

Eliminating l, m, n between (2) and (3), we see that the normals, in question, lie on the surface

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0,$$

which is a quadric cone.

2. *Lines are drawn from the centre of the quadric*

$$ax^2+by^2+cz^2=1,$$

proportional to the area of the perpendicular central section ; show that the locus of their extremities is a quadric

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=constant.$$

Consider any central plane section

$$lx+my+nz=0. \quad ...(1)$$

The area of the conic in which this plane cuts the given quadric is

$$=\pi/p\sqrt{abc}=A,$$

where, p, the length of the perpendicular from the origin to the tangent plane parallel to the plane (1) is given by

$$p=\frac{\sqrt{\left(\Sigma\frac{l^2}{a}\right)}}{\sqrt{\Sigma l^2}}=\sqrt{\left(\Sigma\frac{l^2}{a}\right)},$$

where we have supposed that l, m, n are actual direction cosines.

We require the locus of the point (x, y, z) where

$$x=lAk,\ y=mAk,\ z=nAk\ ;$$

k being the constant of proportionality.

$$\therefore\quad x=\frac{l\pi}{\sqrt{\left(\Sigma\frac{l^2}{a}\sqrt{abc}\right)}}\ k=\frac{l}{\sqrt{\left(\Sigma\frac{l^2}{a}\right)}}k',\ \text{etc.},$$

where $k'=\pi k\sqrt{abc}$.

$$\therefore\qquad \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=k'^2.$$

Hence the result.

3. *Show that the axes of the sections of the surface*

$$ax^2+by^2+cz^2=1,$$

which pass through the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

lie on the cone

$$\frac{(b-c)(mz-ny)}{x}+\frac{(c-a)(nx-lz)}{y}+\frac{(a-b)(ly-mx)}{z}=0.$$

Let

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}, \quad \text{.. } (i)$$

$$\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}, \quad \text{...}(ii)$$

be the principal axes of any section through the given line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}. \quad \text{...}(iii)$$

The axes being perpendicular to each other, we have

$$l_1l_2+m_1m_2+n_1n_2=0. \quad \text{...}(iv)$$

Also as in § 9·24, page 182, we have

$$al_1l_2+bm_1m_2+cn_1n_2=0. \quad \text{...}(v)$$

Also the lines (i), (ii) and (iii) are coplanar. Therefore

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l, & m, & n \end{vmatrix}=0, \quad \text{...}(vi)$$

or

$$l_1(m_2n-mn_2)+m_1(n_2l-nl_2)+n_1(l_2m-lm_2)=0.$$

Eliminating l_1, m_1, n_1 from (iv), (v) and (vi), we have

$$\begin{vmatrix} l_2, & m_2, & n_2 \\ al_2, & bm_2, & cn_2 \\ m_2n-mn_2, & n_2l-nl_2, & l_2m-lm_2 \end{vmatrix}=0. \quad \text{.. } (vii)$$

Now, eliminating l_2, m_2, n_2 from (ii) and (vii), we obtain the locus as required.

4. *One axis of a central section of the conicoid*

$$ax^2+by^2+cz^2=1,$$

lies in the plane

$$ux+vy+wz=0.$$

Show that the other lies on the cone

$$(b-c)uyz+(c-a)vzx+(a-b)wxy=0$$

(C.U. 1914)

Let

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}\ ,\ \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2},$$

be the two axes of a central section such that the second lies in the given plane for which we have the condition

$$ul_2+vm_2+wn_2=0. \qquad ...(i)$$

Also, as in § 9·24,

$$l_1l_2+m_1m_2+n_1n_2=0 \qquad ...(ii)$$

$$al_1l_2+bm_1m_2+cn_1n_2=0. \qquad ...(iii)$$

Eliminating l_2, m_2, n_2 from (i), (ii) and (iii), we have

$$\begin{vmatrix} u, & v, & w \\ l_1, & m_1, & n_1 \\ al_1, & bm_1, & cn_1 \end{vmatrix}=0,$$

or

$$um_1n_1(b-c)+vn_1l_1(c-a)+wl_1m_1(a-b)=0.$$

With the help of this condition, we see that the locus of the axis

$$x/l_1=y/m_1=z/n_1$$

is the cone

$$(b-c)uyz+(c-a)vzx+(a-b)wxy=0.$$

Exercises

1. Show that the section of the ellipsoid

$$9x^2+6y^2+14z^2=3,$$

by the plane

$$x+y+z=0,$$

is an ellipse with semi-axes $1/2$ and $\sqrt{(9/22)}$. Also obtain their equations.

[*Ans.* $\frac{1}{2}x=y=-\frac{1}{3}z\ ;\ x/4=-y/5=z.$

2. Show that the curve

$$x^2+7y^2-10z^2+9=0,\ x+2y+3z=0$$

is a hyperbola whose transverse axis is 6 and the direction cosines of whose axes are proportional to (6, 3, −4) and (17, −22, 9).

3. A_1, A_2, A_3 are the areas of three mutually perpendicular central sections of an ellipsoid ; show that $A_1^{-2}+A_2^{-2}+A_3^{-2}$ is constant.

4. Show that all plane sections of

$$ax^2+by^2+cz^2=1$$

which are rectangular hyperbolas and which pass through the point (α, β, γ) touch the cone

$$\frac{(x-\alpha)^2}{b+c}+\frac{(y-\beta)^2}{c+a}+\frac{(z-\gamma)^2}{a+b}=0.$$

5. Any plane whose normal lies on the cone

$$bcx^2+cay^2+abz^2=0,$$

cuts the surface

$$ax^2+by^2+cz^2=1,$$

in a parabola.

6. The director circle of a plane central section of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

has a radius of constant length r. Show that the plane section touches the cone

$$\frac{x^2}{a^2(b^2+c^2-r^2)}+\frac{y^2}{b^2(c^2+a^2-r^2)}+\frac{z^2}{c^2(a^2+b^2-r^2)}=0.$$

7. If a length PQ be taken on the normal at any point P of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

equal in length to $l^2A/\pi abc$ where l is a constant and A is the area of the section of the ellipsoid by the diametral plane of OP, show that the locus of Q is

$$\frac{a^2x^2}{(a^2+l^2)^2}+\frac{b^2y^2}{(b^2+l^2)^2}+\frac{c^2z^2}{(c^2+l^2)^2}=1.$$

8. Prove that if l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction ratios of the principal axes of any plane section of the quadric

$$ax^2+by^2+cz^2=1,$$

then

$$\frac{l_1l_2}{b-c}=\frac{m_1m_2}{c-a}=\frac{n_1n_2}{a-b}.$$

9. Find the equation of the central plane section of the quadric

$$ax^2+by^2+cz^2=1$$

which has one of its axes along the line

$$x/l=y/m=z/n.$$

[*Ans.* $\Sigma[m^2(a-b)+n^2(a-c)]lx=0$.

10. Show that central plane sections of an ellipsoid of constant area touch a quadric cone.

9·3. Axes of non-central plane sections. *To determine the lengths and direction ratios of the section of the central conicoid*

$$ax^2+by^2+cz^2=1, \qquad \ldots(1)$$

by the plane

$$lx+my+nz=p. \qquad \ldots(2)$$

Centre of the plane section, now, is not the origin. If (α, β, γ) is the centre of the section, the plane (2) is also represented by the equation

$$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2,$$

so that we get

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}=k, \text{ (say)}.$$

$$\therefore \qquad \alpha=\frac{lk}{a}, \ \beta=\frac{mk}{b}, \ \gamma=\frac{nk}{c}.$$

Hence

$$k=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}=\frac{k^2}{p}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right),$$

or

$$k=\frac{p}{l^2/a+m^2/b+n^2/c}.$$

If we write

$$p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c},$$

we get

$$\left(\frac{lp}{ap_0^2}, \ \frac{mp}{bp_0^2}, \ \frac{np}{cp_0^2}\right)$$

as the co-ordinates of the centre of the section. The equation of the conicoid referred to this point as origin is

$$a\left(x+\frac{lp}{ap_0^2}\right)^2+b\left(y+\frac{mp}{bp_0^2}\right)^2+c\left(z+\frac{np}{cp_0^2}\right)^2=1,$$

or

$$ax^2+by^2+cz^2+\frac{2p}{p_0^2}(lx+my+nz)+\frac{p^2}{p_0^2}=1. \qquad ...(3)$$

Also, the equation of the plane (2) becomes

$$lx+my+nz=0. \qquad ...(4)$$

Now the conic

$$ax^2+by^2+cz^2+\frac{2p}{p_0^2}(lx+my+nz)=1-\frac{p^2}{p_0^2},\ lx+my+nz=0, \qquad ...(5)$$

is the same as the conic

$$ax^2+by^2+cz^2=1-\frac{p^2}{p_0^2},\ lx+my+nz=0, \qquad ...(6)$$

for, points whose co-ordinates satisfy the equations (5) also satisfy the equations (6).

Putting

$$1-\frac{p^2}{p_0^2}=d^2,$$

and replacing the a, b, c by a/d^2, b/d^2, c/d^2 respectively in the equations (5) and (6) of the previous article, we get

$$\frac{l^2}{\frac{ar^2}{d^2}-1}+\frac{m^2}{\frac{br^2}{d^2}-1}+\frac{n^2}{\frac{cr^2}{d^2}-1}=0, \qquad ...(7)$$

$$\frac{\lambda\left(\frac{ar^2}{d^2}-1\right)}{l}=\frac{\mu\left(\frac{br^2}{d^2}-1\right)}{m}=\frac{\nu\left(\frac{cr^2}{d^2}-1\right)}{n}, \qquad ...(8)$$

which give the *lengths* r_1, r_2 and the *direction ratios* l, m, n respectively at the corresponding semi-axes of the section.

9·31. Area of the plane section. If the section be an ellipse, we have its area

$$=\pi r_1 r_2$$
$$=\pi d^2\sqrt{\left(\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}\right)}$$
$$=\pi\left(1-\frac{p^2}{l^2/a+m^2/b+n^2/c}\right)\sqrt{\left(\frac{l^2+m^2+n^2}{bcl^2+cam^2+abn^2}\right)}.$$

9·32. Parallel plane sections. Comparing the equations (7) and (8) with the equations (5) and (6) of the previous article, we see that if α, β be the lengths of the semi-axes of the section by the central plane

$$lx+my+nz=0, \qquad ...(9)$$

then the semi-axes of the section by the parallel plane

$$lx+my+nz=p \qquad \ldots(10)$$

are $d\alpha$ and $d\beta$,

or

$$\alpha\sqrt{\left(1-\frac{p^2}{p_0{}^2}\right)} \text{ and } \beta\sqrt{\left(1-\frac{p^2}{p_0{}^2}\right)}$$

and the corresponding axes are parallel.

Thus we see that *parallel plane sections of a central conicoid are similar and similarly situated conics.*

Again, if A_0 and A are the areas of the sections by the planes (9) and (10), we have

$$A_0=\pi\alpha\beta,$$

and

$$A=\pi d^2\alpha\beta=A_0\left(1-\frac{p^2}{p_0{}^2}\right).$$

Thus

$$\frac{A}{A_0}=\left(1-\frac{p^2}{\Sigma l^2/a}\right).$$

Note. p/p_0 can easily be seen to be the ratio of the lengths of the perpendiculars from the centre to the given plane and to the parallel tangent plane.

Examples

1. *Show that the area of the section of an ellipsoid by a plane which passes through the extremities of three conjugate semi-diameters is in a constant ratio to the area of the parallel central section.*

Consider the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the co-ordinates of the extremities of three conjugate semi-diameters of the ellipsoid. The equation of the plane PQR is

$$\frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1. \qquad \ldots(1)$$

The central plane parallel to (1) is

$$\frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=0. \qquad \ldots(2)$$

Re-writing these equations as

$$lx+my+nz=1,\ lx+my+nz=0,$$

we see that the ratio of the areas of the two sections

$$=\left(1-\frac{1}{\Sigma a^2l^2}\right) \qquad (\S\ 9{\cdot}32)$$

Again

$$\Sigma a^2l^2=\Sigma\frac{(x_1+x_2+x_3)^2}{a^2}=3,$$

making use of relations C, D of § 8·8.

Hence the result.

2. *Find the angle between the asymptotes of the conic*

$$ax^2+by^2+cz^2=1,\ lx+my+nz=p.$$

Let θ be the required angle.

If r_1^2, r_2^2 be the squares of the semi-axes of the conic, we have

$$\tan\frac{\theta}{2}=\sqrt{\frac{-r_2^2}{r_1^2}},$$

or

$$\tan^2\theta=\frac{-4r_1^2r_2^2}{(r_1^2+r_2^2)^2}$$

$$=\frac{-4(l^2+m^2+n^2)(bcl^2+cam^2+abn^2)}{[(b+c)l^2+(c+a)m^2+(a+b)n^2]^2}.$$

Exercises

1. Find the lengths and directions of the axes of the section of the ellipsoid $9x^2+6y^2+14z^2=3$ by the plane $x+y+z=1$.

[*Ans.* $\frac{3}{2\sqrt{2}}$, $\sqrt{(22)}44$, (4, −5, 1), (2, 1, −3).

2. Show that the plane $x+y+z=1$ cuts the quadric

$$11x^2-13y^2-4z^2=5,$$

in a hyperbola and find the direction ratios of its axes.

[*Ans.* −3, 1, 2 ; 1, −5, 4.

3. Show that the plane $x+2y+3z=4$ cuts the conicoid

$$2x^2+y^2-2z^2=1$$

in a parabola, the direction cosines of whose axis are proportional to 1, 4, −3.

4. The ellipsoid $x^2+2y^2+3z^2=1$ is cut by parallel planes

$$2x+3y+4z=2,\ 2x+3y+4z=3\ ;$$

show that the areas of the sections made by the planes are in the ratio 59 : 29.

5. Find the locus of the centres of the sections of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

which are of constant area πk^2,

$$\left[Ans.\ a^2b^2c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)^2=k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right).\right.$$

9·4. Circular Sections. *To determine the circular sections of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \ldots(1)$$

Writing the equation of the ellipsoid in the form

$$\frac{1}{a^2}(x^2+y^2+z^2-a^2)+y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0,$$

we see that the two planes

$$y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)=0, \qquad \ldots(2)$$

meet the ellipsoid where they meet the sphere

$$x^2+y^2+z^2=a^2,$$

but as a plane necessarily cuts a sphere in a circle, we find that the planes (2) cut the ellipsoid (1) in circles.

Similarly, if we re-write the equation (1) in the forms

$$\frac{1}{b^2}(x^2+y^2+z^2-b^2)+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0, \quad \ldots(3)$$

$$\frac{1}{c^2}(x^2+y^2+z^2-c^2)+x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0, \quad \ldots(4)$$

we find that the planes

$$x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0,$$

and

$$x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)=0,$$

cut the ellipsoid in circles.

Thus there are *three* pairs of central planes which cut an ellipsoid in circles.

If $a^2>b^2>c^2$, the second of these equations *only* gives real planes so that in this case the real pair of central planes of circular sections is

$$\frac{x}{a}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0. \quad \ldots(5)$$

Since parallel sections are similar, *the two systems of planes*

$$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=\lambda,$$

and

$$\frac{x}{a}\sqrt{(a^2-b^2)}-\frac{z}{c}\sqrt{(b^2-c^2)}=\mu,$$

which are parallel to those given by the equations (5) *cut the ellipsoid in circles for all values of* λ *and* μ.

9·41. *Any two circular sections of an ellipsoid of opposite systems lie on a sphere.*

Let $$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=\lambda$$

and $$\frac{x}{a}\sqrt{(a^2-b^2)}-\frac{z}{c}\sqrt{(b^2-c^2)}=\mu,$$

be the equations of the planes of any two circular sections of opposite systems.

The conicoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1+k\left(\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}-\lambda\right)\times$$
$$\left(\frac{x}{a}\sqrt{(a^2-b^2)}-\frac{z}{c}\sqrt{(b^2-c^2)}-\mu\right)=0, \quad \ldots(1)$$

which passes through the two circular sections for all values of k, will represent a sphere, if k can be chosen so that

$$\frac{1}{a^2}+\frac{k(a^2-b^2)}{a^2}=\frac{1}{b^2}=\frac{1}{c^2}-\frac{k(b^2-c^2)}{c^2}.$$

Now, $$k=1/b^2,$$

clearly satisfies these two equations.

Substituting this value of k in (1), we get

$$x^2+y^2+z^2-\frac{(\lambda+\mu)\sqrt{(a^2-b^2)}}{a}x+\frac{(\lambda-\mu)\sqrt{(b^2-c^2)}}{c}z+\lambda\mu-b^2=0,$$

which represents the sphere through the two circular sections.

Hence the proposition is proved.

Exercises

1. Show that the real central circular sections of the hyperboloids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ and } \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

are given by the planes

$$\frac{y}{b}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(a^2+c^2)}=0 \text{ and } \frac{x}{a}\sqrt{(a^2+b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0.$$

Also show that any two circular sections of opposite systems in the case of either hyperboloid lie on a sphere.

2. Find the real circular sections of the following conicoids :

(*i*) $2x^2+11y^2+z^2=1$. [*Ans.* $3y+z=\lambda$, $3y-z=\mu$.

(*ii*) $10x^2-2y^2+z^2+2=0$. [*Ans.* $\sqrt{3}x+y=\lambda$, $\sqrt{3}x-y=\mu$.

(*iii*) $15x^2-y^2-10z^2+4=0$. [*Ans.* $4x+3z=\lambda$, $4x-3z=\mu$.

3. Find the equation of the sphere which contains the two circular sections of the ellipsoid $x^2-3y^2+2z^2=4$ through the point (1, 2, 3).

[*Ans.* $x^2+y^2+z^2-16y+6z+7=0$.

4. Find the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=\lambda$$

cuts the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

[*Ans.* $b\sqrt{[1-\lambda^2/(a^2-c^2)]}$.

[*Hint.* Obtain the equation of the sphere which passes through the given circle and any circle of the opposite system and determine the radius of the circle in which the given plane cuts it.]

5. Show that the circular-sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

passing through one extremity of X-axis are both of radius, r, where

$$\frac{r^2}{b^2}=\frac{b^2-c^2}{a^2-c^2}.$$

6. Prove that the radius of a circular section of the ellipsoid at a distance p from the centre is $b\sqrt{(1-p^2b^2/a^2c^2)}$.

7. Show that the locus of the centres of the spheres which pass through the origin and cut the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

in a pair of real circles is the hyperbola

$$\frac{a^2x^2}{a^2-b^2}-\frac{c^2z^2}{b^2-c^2}=b^2,\ y=0.$$

8. If p_1, p_2, p_3 be the lengths of the perpendiculars from the extremities P_1, P_2, P_3 of conjugate semi-diameters on one of the planes of central circular sections of the ellipsoid,

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

then show that

$$p_1^2+p_2^2+p_3^2=a^2c^2/b^2. \qquad (B.U.\ 1931)$$

9. A cone is drawn with its vertex at the centre of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and its base is a circular section of the ellipsoid. If the cone contains three mutually perpendicular generators, prove that the distance of the section from the centre of the ellipsoid is

$$\frac{abc}{\sqrt{(b^2c^2+c^2a^2+a^2b^2)}}.$$

9·42. Umbilics.

Def. A point on a quadric such that the planes parallel to the tangent plane at the point determine circular sections on the surface is called an *umbilic*.

Clearly, *umbilic* is a point-circle which lies on a quadric.

The umbilics are the extremities of the diameters which pass through the centres of the systems of circular sections.

To determine the real umbilics of the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

If f, g, h be an umbilic, the tangent plane

$$\frac{fx}{a^2}+\frac{gy}{b^2}+\frac{hz}{c^2}=1,$$

at the point is parallel to either of the central circular sections

$$\frac{x}{a}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0.$$

$$\therefore\ g=0 \text{ and } \frac{f}{a\sqrt{(a^2-b^2)}}=\pm\frac{h}{c\sqrt{(b^2-c^2)}}.$$

But

$$\frac{f^2}{a^2}+\frac{g^2}{b^2}+\frac{h^2}{c^2}=1.$$

Hence

$$f=\pm\frac{a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}},\quad g=0,\quad h=\pm\frac{c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}}.$$

These are the co-ordinates of the four real umbilics.

Exercises

1. Show that the hyperboloid of one sheet has no real umbilics.

2. Find the real umbilics of the hyperboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

$$[Ans.\ \frac{\pm a\sqrt{(a^2+b^2)}}{\sqrt{(a^2+c^2)}},\ 0,\ \frac{\pm c\sqrt{(b^2-c^2)}}{\sqrt{(a^2+c^2)}}.$$

3. Find the umbilics of the ellipsoid $2x^2+3y^2+6z^2=6$.

$$[Ans.\ (\pm\tfrac{1}{2}\sqrt{6},\ 0,\ \pm\tfrac{1}{2}\sqrt{2}).$$

4. Show that the four real umbilics of an ellipsoid lie upon a circle.

5. Prove that the perpendicular distance from the centre to the tangent plane at an umbilic of the ellipsoid is ac/b. *(U.P. 1937)*

9·5. Sections of paraboloids. *To determine the nature of the section of the paraboloid*

$$ax^2+by^2=2cz,$$

by the plane

$$lx+my+nz=p.$$

Let $l\neq 0$ so that the plane is not perpendicular to the YZ plane. As in § 9·3, the equations of the projection of the section on the YZ plane are

$$x=0,$$

$$(am^2+bl^2)y^2+2amnyz+an^2z^2-2apmy-2(apn+cl^2)z+ap^2=0.$$

The projection and, therefore, also the section is an ellipse, parabola, or hyperbola according as

$$a^2m^2n^2-an^2(am^2+bl^2)\begin{cases}<\\ =0\\ >\end{cases}\quad\text{or}\quad abn^2l^2\begin{cases}>\\ =0.\\ <\end{cases}$$

Thus for a parabola $n=0$. If $n\neq 0$, the section will be an ellipse or hyperbola according as ab is positive or negative that is according as the paraboloid is elliptic or hyperbolic.

If $l=0$ and $m\neq 0$ then, by projecting on the XZ plane, we get a similar result.

If $l=m=0$ then n cannot be equal to zero and the section is then clearly an ellipse or hyperbola according as ab is positive or negative.

Thus we have proved that *all the sections of a paraboloid which are parallel to the axis of the surface are parabolas : all other sections of an elliptic paraboloid are ellipses and of an hyperbolic paraboloid are hyperbolas.*

9·51. Axes of plane sections of paraboloids. *To determine the lengths and the direction ratios of the section of the paraboloid*

$$ax^2+by^2=2cz, \qquad \ldots(1)$$

by the plane

$$lx+my+nz=p. \qquad \ldots(2)$$

Let (α, β, γ) be the centre of the section so that the plane (2) is also represented by the equation

$$a\alpha x+b\beta y-cz=a\alpha^2+b\beta^2-c\gamma.$$

Comparison gives

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{-c}{n}=\frac{a\alpha^2+b\beta^2-c\gamma}{p}.$$

Therefore $$\alpha=-\frac{lc}{an},\quad \beta=-\frac{mc}{bn},$$

and $$c\gamma=a\alpha^2+b\beta^2+\frac{pc}{n}=\frac{c^2}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}\right).$$

If we write

$$k=\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c},$$

we find that the centre of the section is

$$\left(-\frac{lc}{an},\ -\frac{mc}{bn},\ \frac{kc}{n^2}\right).$$

The equation of the paraboloid referred to this point as the origin is

$$a\left(x-\frac{lc}{an}\right)^2+b\left(y-\frac{mc}{bn}\right)^2=2c\left(z+\frac{kc}{n^2}\right),$$

or $$ax^2+by^2-\frac{2c}{n}(lx+my+nz)-\frac{c(kc+np)}{n^2}=0.$$

Also, the equation of the plane (2) now becomes

$$lx+my+nz=0.$$

Now the conic

$$\left.\begin{aligned} ax^2+by^2-\frac{2c}{n}(lx+my+nz)-\frac{c(kc+np)}{n^2}&=0\\ lx+my+nz&=0\end{aligned}\right\} \quad \ldots(3)$$

is the same as the conic

$$ax^2+by^2=\frac{c(kc+np)}{n^2},\ lx+my+nz=0.$$

Let us write

$$p_0{}^2=c(kc+np)=c\left(\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right).$$

The semi-diameters of length r of the conicoid

$$ax^2+by^2=\frac{p_0{}^2}{n^2},$$

are the generators of the cone

$$ax^2+by^2=\frac{p_0{}^2}{n^2}\cdot\frac{x^2+y^2+z^2}{r^2},$$

i.e., $$x^2(an^2r^2-p_0{}^2)+y^2(bn^2r^2-p_0{}^2)-z^2p_0{}^2=0 \quad \ldots(4)$$

The plane

$$lx+my+nz=0,$$

will touch this cone if

$$\frac{l^2}{an^2r^2-p_0{}^2}+\frac{m^2}{bn^2r^2-p_0{}^2}-\frac{n^2}{p_0{}^2}=0, \quad \ldots(5)$$

or

$$abn^6r^4-n^2r^2p_0{}^2[(a+b)n^2+am^2+bl^2]+p_0{}^4(l^2+m^2+n^2)=0,$$

which is a quadratic equation in r^2 and has two roots r_1^2, r_2^2, which are the squares of the semi-axes of the section.

Also, if λ, μ, ν be the direction ratios, of the axis of length $2r$, the plane (2) touches the cone (4) along the line

$$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}$$

and is, therefore, identical with

$$(an^2r^2-p_0^2)\lambda x+(bn^2r^2-p_0^2)\mu y-\nu p_0^2 z=0,$$

so that we have

$$\frac{(an^2r^2-p_0^2)\lambda}{l}=\frac{(bn^2r^2-p_0^2)\mu}{m}=\frac{-p_0^2\nu}{n}, \qquad \ldots(6)$$

which determine the direction ratios of the axis of length $2r$; r being given from the equation (5).

9·52. The section will be rectangular hyperbola, if

$$r_1^2+r_2^2=0.$$

This requires

$$\mathbf{(a+b)n^2+am^2+bl^2=0.}$$

Ex. Obtain the conclusion of § 9·5 with the help of equation (5) of this article.

9·53. Area of the section.

If the section be elliptic, its area

$$=\pi r_1 r_2$$
$$=\frac{\pi p_0^2}{n^3}\sqrt{\left[\frac{l^2+m^2+n^2}{ab}\right]}$$
$$=\frac{\pi c}{n^3}\left[\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right]\sqrt{\left[\frac{l^2+m^2+n^2}{ab}\right]}.$$

9·54. If θ be the angle between the asymptotes of the section, then as in § Ex. 2, page 189.

$$\tan^2\theta=\frac{-4r_1^2r_2^2}{(r_1^2+r_2^2)^2}=\frac{-4abn^2(l^2+m^2+n^2)}{[(a+b)n^2+am^2+bl^2]^2},$$

which being independent of p, we deduce that the angle between the asymptotes of parallel plane sections is the same.

Thus we see *that parallel plane sections of a paraboloid are similar.*

Exercises

1. Show that the section of the paraboloid

$$ax^2+by^2=2cz,$$

by a tangent plane to the cone

$$\frac{x^2}{b}+\frac{y^2}{a}+\frac{z^2}{a+b}=0,$$

is a rectangular hyperbola.

2. Prove that the axis of the section of the conicoid $ax^2+by^2=2z$ by the plane $lx+my+nz=0$ lie on the cone

$$\frac{bl}{x}-\frac{am}{y}+\frac{(a-b)n}{z}=0.$$

3. If the area of the section of

$$ax^2+by^2=2cz,$$

be constant and equal to πk^2, the locus of the centre is

$$(a^2x^2+b^2y^2+c^2)(ax^2-by^2-2cz)^2=abc^2k^4.$$

9·6. Circular sections of paraboloids. *To determine the circular sections of the paraboloid*

$$ax^2+by^2=2cz. \qquad \ldots(1)$$

The equation (1) can be written in the forms

$$a\left(x^2+y^2+z^2-\frac{2cz}{a}\right)+y^2(b-a)-az^2=0,$$

$$b\left(x^2+y^2+z^2-\frac{2cz}{b}\right)+x^2(a-b)-bz^2=0.$$

Therefore, as before, the two pairs of planes

$$y^2(b-a)-az^2=0, \qquad \ldots(2)$$

and

$$x^2(a-b)-bz^2=0, \qquad \ldots(3)$$

determine circular sections through the origin.

If a or b is negative and the other positive, neither of the equations (2) and (3) gives real planes.

Hence *hyperbolic paraboloids have no real circular sections.*

Of the two pairs of planes (2) and (3), one will be real if a and b are of the same sign.

In case $a>b>0$,

$$x^2(a-b)-bz^2=0,$$

gives real circular sections through the origin and the two real systems of circular sections are given by

$$x\sqrt{(a-b)}+\sqrt{bz}=\lambda,\ x\sqrt{(a-b)}-\sqrt{bz}=\mu.$$

Exercises

1. Show that any two circular sections of opposite systems of an elliptic paraboloid lie on a sphere.

2. Find the real circular sections of the paraboloid :

(*i*) $13y^2+4z^2=2x.$ [*Ans.* $2x\pm3y=\lambda.$

(*ii*) $x^2+5z^2+4y=0,$ [*Ans.* $y\pm2z=\lambda.$

9·61. Umbilics of a paraboloid. *To determine the umbilics of the paraboloid*

$$ax^2+by^2=2cz,\ a>b>0.$$

Circular sections are determined by the planes

$$x\sqrt{(a-b)}+\sqrt{bz}=\lambda,\ x\sqrt{(a-b)}-\sqrt{bz}=\mu.$$

If f, g, h be an umbilic, the tangent plane

$$afx+bgy-c(z+h)=0$$

thereat is parallel to either of the circular sections.

$$\therefore \qquad g=0 \text{ and } f=\pm\frac{c}{a}\sqrt{\left(\frac{a-b}{b}\right)}.$$

Also,

$$af^2+bg^2=2ch.$$

Therefore,

$$h=\frac{(a-b)c}{2ab}.$$

Hence $\left[\pm\frac{c}{a}\sqrt{\left(\frac{a-b}{b}\right)},\ 0,\ \frac{(a-b)c}{2ab}\right]$

are the two real umbilics of the paraboloid.

Ex. 1. Find the umbilics of the paraboloids

(*i*) $4x^2+5y^2=40z$. (*ii*) $25x^2+16y^2=2z$.

[*Ans.* (*i*) $(0, \pm 2, \frac{1}{2})$; (*ii*) $(\pm 3/100, 0, 9/800)$.

CHAPTER X

GENERATING LINES OF CONICOIDS

10·1. Generating lines of the hyperboloid of one sheet. We re-write the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad \ldots(1)$$

of a hyperboloid of one sheet in the form

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2},$$

or
$$\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=\left(1-\frac{y}{b}\right)\left(1+\frac{y}{b}\right).$$

This may again be written in either of the two forms

$$\frac{\frac{x}{a}-\frac{z}{c}}{1-\frac{y}{b}}=\frac{1+\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}} \qquad \ldots(2)$$

or
$$\frac{\frac{x}{a}-\frac{z}{c}}{1+\frac{y}{b}}=\frac{1-\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}}. \qquad \ldots(3)$$

We consider, now, the two *families* of lines obtained by putting the equal fractions (2) and (3) equal to *arbitrary constants* λ and μ respectively.

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right), \qquad \ldots(A)$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right). \qquad \ldots(B)$$

To each value of the constant λ, corresponds a member of the family of lines (A) and to each value of the constant μ corresponds a member of the family of lines (B).

Now it will be shown that *every point of each of the lines* (A) *and* (B) *lies on the hyperboloid* (1).

If (x_0, y_0, z_0) be *any* point of a member of the family (A) obtained for some value λ_0 of λ_1 we have

$$\frac{x_0}{a}-\frac{z_0}{c}=\lambda_0\left(1-\frac{y_0}{b}\right),\ 1+\frac{y_0}{b}=\lambda_0\left(\frac{x_0}{a}+\frac{z_0}{c}\right).$$

On eliminating λ_0 from these, we obtain

$$\frac{x_0^2}{a^2}-\frac{z_0^2}{c^2}=1-\frac{y_0^2}{b^2} \quad \text{or} \quad \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1,$$

which relation shows that (x_0, y_0, z_0) is a point of the hyperboloid (1).

A similar proof holds for the family of lines (B).

Thus as λ and μ vary, we get two families of lines (A) and (B) each member of each of which lies wholly on the hyperboloid. These two families of lines are called *two systems of generating lines (or generators) of the hyperboloid.*

We shall now proceed to discuss some properties of these systems of generating lines.

10·11. *Through every point of the hyperboloid there passes one generator of each system.*

Let (x_0, y_0, z_0) be any point of the hyperboloid so that we have

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1. \qquad ...(4)$$

Now the line

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right),$$

will pass through the point (x_0, y_0, z_0) if, and only if, λ has a value equal to each of the two fractions

$$\left(\frac{x_0}{a}-\frac{z_0}{c}\right)\Big/\left(1-\frac{y_0}{b}\right),\ \left(1+\frac{y_0}{b}\right)\Big/\left(\frac{x_0}{a}+\frac{z_0}{c}\right). \qquad ...(5)$$

Now, by virtue of the relation (4), these two fractions are equal.

Thus the member of the system (A) corresponding to either of the equal values (5) of λ will pass through the given point (x_0, y_0, z_0). Similarly it can be shown that the member of the system (B) corresponding to either of the equal values

$$\left(\frac{x_0}{a}-\frac{z_0}{b}\right)\Big/\left(1+\frac{y_0}{b}\right),\ \left(1-\frac{y_0}{b}\right)\Big/\left(\frac{x_0}{a}+\frac{z_0}{c}\right)$$

of, μ, passes through the given point (x_0, y_0, z_0).

10·12. *No two generators of the same system intersect.*

Let

$$(i)\ \frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right),\quad (ii)\ 1+\frac{y}{b}=\lambda_1\left(\frac{x}{a}+\frac{z}{c}\right),$$

$$(iii)\ \frac{x}{a}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right),\quad (iv)\ 1+\frac{y}{b}=\lambda_2\left(\frac{x}{a}+\frac{z}{c}\right),$$

be any two different generators of the λ system.

Subtracting (iii) from (i), we obtain

$$(\lambda_1-\lambda_2)\left(1-\frac{y}{b}\right)=0 \text{ or } y=b, \text{ for } \lambda_1\neq\lambda_2,$$

Again, from (ii) and (iv), we obtain

$$\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_2}\right)\left(1+\frac{y}{b}\right)=0 \text{ or } y=-b, \text{ for } \lambda_1\neq\lambda_2,$$

Thus we see that these four equations are inconsistent and accordingly the two lines do not intersect.

10·13. *Any two generators belonging to different systems intersect.* Let

$$(i)\ \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\quad (ii)\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right),$$

$$(iii)\ \frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\quad (iv)\ 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right),$$

be two generators, one of each system.

Firstly, we solve simultaneously the equations (i), (ii) and (iii). Now (i) and (iii) give

$$\lambda\left(1-\frac{\lambda}{b}\right)=\mu\left(1+\frac{y}{b}\right) \text{ or } y=b\,\frac{\lambda-\mu}{\lambda+\mu}.$$

Substituting this value of y in (i) and (ii), we obtain

$$\frac{x}{a}-\frac{z}{c}=\frac{2\lambda\mu}{\lambda+\mu},\quad \frac{x}{a}+\frac{z}{c}=\frac{2}{\lambda+\mu}.$$

These give, on adding and subtracting,

$$x=a\,\frac{1+\lambda\mu}{\lambda+\mu},\ z=c\,\frac{1-\lambda\mu}{\lambda+\mu}.$$

Now, as may easily be seen, these values of x, y, z satisfy (iv) also. Thus the two lines intersect and the point of intersection is

$$\left(a\,\frac{1+\lambda\mu}{\lambda+\mu},\ b\,\frac{\lambda-\mu}{\lambda+\mu},\ c\,\frac{1-\lambda\mu}{\lambda+\mu}\right) \qquad \ldots(6)$$

Another method. The planes

$$\frac{x}{a}-\frac{z}{c}-\lambda\left(1-\frac{y}{b}\right)-k\left[1+\frac{y}{b}-\lambda\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0,$$

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)-k'\left[1-\frac{y}{b}-\mu\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0,$$

pass through the two lines respectively for all values of k and k'.

Now, obviously these equations become identical for

$$k=\mu \text{ and } k'=\lambda.$$

Thus the two lines are coplanar and as such they intersect. Also the plane through the two lines, obtained by putting $k=\mu$ or $k'=\lambda$ is

$$\frac{1+\lambda\mu}{\lambda+\mu}\cdot\frac{x}{a}+\frac{\lambda-\mu}{\lambda+\mu}\cdot\frac{y}{b}-\frac{1-\lambda\mu}{\lambda+\mu}\cdot\frac{z}{c}=1. \qquad \ldots(7)$$

Cor. 1. Now the plane (7) through two generators of the opposite systems is the tangent plane to the hyperboloid (1) at the point of intersection (6) of the two generators. Since also through every point of the hyperboloid there pass two generators, one of each system, we see that *the tangent plane at a point of hyperboloid meets the hyperboloid in the two generators through the point.*

Cor. 2. *Any plane through a generating line is the tangent plane at some point of the generator.* Now like every plane section, the section of the hyperboloid by any plane through a generator is a

conic of which the given generator is a part. Thus the conic is degenerate and the residue must also be a line. At the point of intersection of the lines constituting this degenerate plane section, the plane will touch the hyperboloid.

Ex. Prove this result analytically also.

Cor. 3. Parametric Equations of the hyperboloid. The co-ordinates (6) show that

$$x=a\frac{1+\lambda\mu}{\lambda+\mu},\quad y=b\frac{\lambda-\mu}{\lambda+\mu},\quad z=c\frac{1-\lambda\mu}{\lambda+\mu}$$

are the parametric equations of the hyperboloid ; λ. μ being the *two* parameters. These co-ordinates satisfy the equation of the hyperboloid for all values of the parameters λ and μ.

Example

Find the lengths of the side of the skew quadrilateral formed by the four generators of the hyperboloid

$$x^2/4+y^2-z^2=49,$$

which pass through the two points (10, 5, 1) (14, 2, −2).

(*D.U., M.A. 1948*)

Re-writing the given equation in the form

$$\left(\frac{x}{2}-z\right)\left(\frac{x}{2}+z\right)=(7-y)(7+y),$$

we see that the equations of the two systems of generating lines of the hyperboloid are

$$\frac{x}{2}-z=\lambda(7-y),\quad \lambda\left[\frac{x}{2}+z\right]=7+y\ ; \qquad \ldots(i)$$

$$\frac{x}{2}-z=\mu(7+y),\quad \mu\left[\frac{x}{2}+z\right]=7-y. \qquad \ldots(ii)$$

The generators (*i*) and (*ii*) pass through the points

(10, 5, 1) and (14, 2, −2)

for

$$\lambda=2,\ \mu=\tfrac{1}{3} \text{ and } \lambda=\tfrac{2}{5},\ \mu=1$$

respectively.

The two pairs of generators through the two points, therefore, are

$$\begin{cases}\dfrac{x}{2}-z=2(7-y),\ 2\left[\dfrac{x}{2}+z\right]=7+y & \ldots\ (iii)\\[2ex] \dfrac{x}{2}-z=\tfrac{1}{3}(7+y),\ \tfrac{1}{3}\left[\dfrac{x}{2}+z\right]=7-y & \ldots\ (iv)\end{cases}$$

$$\begin{cases}\dfrac{x}{2}-z=\tfrac{2}{5}(7-y),\ \tfrac{2}{5}\left[\dfrac{x}{2}+z\right]=7+y & \ldots\ (v)\\[2ex] \dfrac{x}{2}-=7+y, \qquad \dfrac{x}{2}+z\ =7-y & \ldots\ (vi)\end{cases}$$

Solving in pairs (*iii*), (*vi*) and (*iv*), (*v*), we see that the two other vertices of the skew quadrilateral formed by the four generators are

$$(14, \tfrac{7}{3}, -\tfrac{7}{3}), \ (\tfrac{21}{2}, \tfrac{77}{16}, \tfrac{21}{16}).$$

The lengths of the sides are now easily seen to be

$$\sqrt{(98)}/16, \ \sqrt{(308)}/3, \ \sqrt{2}/3, \ \sqrt{(7970)}/16.$$

Exercises

1. Write down the equations of the two systems of generating lines of the following hyperboloid and determine the pair of lines of the systems which pass through the given point.

(*i*) $x^2+9y^2-z^2=9$, $(3, 1/3, -1)$.

(*ii*) $x^2/9-y^2/16+z^2/4=1$, $(-1, 4/3, 2)$.

[*Ans.* (*i*) $x+3\mu y-z=3\lambda$, $\lambda x-3y+\lambda z=3$; $x+6y-z=6$, $2x-3y+2z=3$

$x-3\mu y-z=3\mu$, $\mu x+3y+\mu z=3$; $x-3y-z=3$, $x+3y+z=3$

(*ii*) $4x-3y+6\lambda z=12\lambda$, $4\lambda x+3\lambda y-6z=12$; $z=2$, $4x+3y=0$

$4x-3y-6\mu z=12\mu$, $4\mu x+3\mu y+6z=12$;

$4x-3y+2z+4=0$, $4x+3y-8z+36=0$.

10·2. *To find the equations of the two generating lines through any point* $(a\cos\theta, b\sin\theta, 0)$, *of the principal elliptic section*

$$x^2/a^2+y^2/b^2=1, \ z=0,$$

of the hyperboloid by the plane $z=0$.

Let

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}$$

be any generator through the point

$$(a\cos\theta, \ b\sin\theta, \ 0).$$

The point

$$(lr+a\cos\theta, \ mr+b\sin\theta, \ nr)$$

on the generator is a point of the hyperboloid for all values of r. Thus the equation

$$\frac{(lr+a\cos\theta)^2}{a^2}+\frac{(mr+b\sin\theta)^2}{b^2}-\frac{n^2r^2}{c^2}=1.$$

i.e.,

$$\left[\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right]r^2+2r\left[\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right]=0,$$

in r must be an identity. This will be so if

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0,$$

and

$$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0.$$

These give

$$\frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta}=\frac{n}{\pm c}.$$

Thus we obtain

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c} \qquad \ldots(C)$$

as the two required generators.

Note. Since every generator of either system meets the plane $z=0$ at a point of the principal elliptic section, we see that the two systems of lines obtained from (C) as θ varies from 0 to 2π are the two systems of generators of the hyperboloid. The form (C) of the equations of two systems of generators is often found more useful than the forms (A) and (B) obtained in § 10·1.

Ex. Show that (A) and (B) are equivalent to (C) for

$$\lambda=\tan(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta),\ \mu=\cot(\tfrac{1}{4}\pi-\tfrac{1}{2}\theta).$$

10·3. *To show that the projections of the generators of a hyperboloid on any principal plane are tangents to the section of the hyperboloid by the principal plane.*

Consider any generator

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}.$$

The equation

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}$$

represents the plane through the generator perpendicular to the XOY plane so that the projection of the generator on the XOY plane is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta},\ z=0$$

or

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\ z=0,$$

which is clearly a tangent line to the section

$$x^2/a^2+y^2/b^2=1,\ z=0$$

of the hyperboloid by the principal plane $z=0$ at the point

$$(a\cos\theta,\ b\sin\theta,\ 0).$$

Again

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{z}{c},$$

is the plane through the generator perpendicular to the XOZ plane so that the projection of the generator on the XOZ plane is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{z}{c},\ y=0$$

or

$$\frac{x\sec\theta}{a}-\frac{z}{c}\tan\theta=1,\ y=0$$

which is clearly the tangent to the section

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1,\ y=0$$

of the hyperboloid by the principal plane $y=0$ at the point

$$(a\sec\theta,\ 0,\ c\tan\theta).$$

Similarly we may show that the projections of the generators on the principal plane $x=0$ touch the corresponding section.

Example

Show that points of intersection R, S of the generators of opposite systems drawn through the points

$$(a\cos\theta,\ b\sin\theta,\ 0),\ (a\cos\phi,\ b\sin\phi,\ 0)$$

of the principal elliptic section of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

are

$$\left(a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\ b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\ \pm c\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta-\phi)}\right).$$

The question can, of course, be solved by solving simultaneously the equations of the generators obtained in § 10·2, but we shall give another method which is perhaps simpler.

Let R (x_1, y_1, z_1) be either of the two points of intersection of the generators.

The tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$$

at R meets the plane $z=0$ of the principal elliptic section in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0,\ z=0$$

which is the line joining the points P, Q whose equation is known to be

$$\frac{x\cos\frac{1}{2}(\theta+\phi)}{a}+\frac{y\sin\frac{1}{2}(\theta+\phi)}{b}=\cos\tfrac{1}{2}(\theta-\phi),\ z=0.$$

Comparing these equations, we obtain

$$x_1=a\frac{\cos\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},\ y_1=b\frac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta-\phi)},$$

Also we have

$$x_1^2/a^2+y_1^2/b^2-z_1^2/c^2=1.$$

Substituting these values of x_1 and y_1 in this relation, we obtain

$$z_1=\pm c\tan\tfrac{1}{2}(\theta-\phi)=\pm c\frac{\sin\frac{1}{2}(\theta-\phi)}{\cos\frac{1}{2}(\theta-\phi)}.$$

Exercises

1. R, S are the points of intersection of generators of opposite systems drawn at the extremities P, Q of semi-conjugate diameters of the principal elliptic section ; show that

(*i*) the locus of the points R, S are the ellipses

$$x^2/a^2+y^2/b^2=2,\ z=\pm c\ ;$$

(*ii*) the perimeter of the skew quadrilateral $PSQR$ taken in order, is constant and equal to $2\ (a^2+b^2+2c^2)$;

(*iii*) $\cot^2\alpha+\cot^2\beta=(a^2+b^2)/c^2$ where

$$\angle RPS=2\alpha \text{ and } \angle RQS=2\beta\ ;$$

(*iv*) the volume of the tetrahedron $PSQR$ is constant and equal to

$$\tfrac{1}{3}abc.$$

2. The generators through a point P on the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

meet the principal elliptic section in points whose eccentric angles differ by a constant 2α ; show that the locus of P is the curve of intersection of the hyperboloid with the cone

$$x^2/a^2+y^2/b^2=z^2/c^2\cos^2\alpha.$$

3. If the generators through a point P on the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

meet the principal elliptic section in two points such that the eccentric angle of one is three times that of the other, prove that P lies on the curve of intersection of the hyperboloid with the cylinder

$$y^2(z^2+c^2)=4b^2z^2.$$

4. Show that the generators through any one of the ends of an equi-conjugate diameter of the principal elliptic section of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

are inclined to each other at an angle of 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other.

[*Ans.* $a^2+b^2=2c^2$.

5. A variable generator of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

intersects generators of the same system through the extremities of a diameter of the principal elliptic section in points P and P' ; show that

$$x_P\ x_{P'}/a^2=y_Py_{P'}/b^2,z_Pz_{P'}=-c^2.$$

6. Show that the shortest distance between generators of the same system drawn at one end of each of the major and minor axes of the principal elliptic section of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

is

$$2abc/\sqrt{(a^2b^2+b^2c^2+c^2a^2)}.$$

7. Show that the shortest distance between generators of the same system drawn at the extremities of the diameters of the principal elliptic section of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

are parallel to the XOY plane and lie on the surfaces

$$abz(x^2+y^2)=\pm(a^2-b^2)cxy.$$

8. Show that the lines through the origin drawn parallel to the line of shortest distance between generators of the same system through the ends of semi-conjugate diameters of the principal elliptic section of the hyperboloid,

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

generate the cone

$$a^2x^2+b^2y^2-2c^2z^2=0.$$

9. A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

in P and P', prove that

$$BP.B'P'=b^2+c^2.$$

10. Q is a point on a generator at any point P of the principal circular section of the hyperboloid

$$c^2(x^2+y^2)-a^2z^2=a^2c^2,$$

such that $PQ=r$; show that the angle between the tangent plane at P and Q is $\tan^{-1}(r/c)$.

11. The generators through a point P on the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1,$$

meet the plane $z=0$ in A, B and the volume of the tetrahedron formed by the generators through A and B is constant and equal to $abc/4$; show that the locus of P is either of the ellipses

$$x^2/a^2+y^2/b^2=4,\ z=\pm\sqrt{3}c.$$

10·4. *To find the locus of the points of intersection of perpendicular generators of the hyperboloid*

$$x^2/a^2+y^2/b^2-z^2/c^2=1. \qquad ...(1)$$

Let (x_1, y_1, z_1) be any point the generators through which are perpendicular.

The generators are the lines in which the tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1 \qquad ...(2)$$

at the point meets the surface. On making (1) homogeneous with the help of (2), we obtain the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}\right)^2. \qquad ...(3)$$

The curve of intersection of (1) and (2) being a pair of lines, the cone with its vertex at the origin and with the curve of intersection of (1) and (2), as the guiding curve, represented by the equation (3), reduces to a pair of planes.

If, l, m, n be the direction ratios of either of the two generators, we have, since they lie on the planes (2) and (3),

$$\frac{lx_1}{a^2}+\frac{my_1}{b^2}-\frac{nz_1}{c^2}=0, \qquad ...(4)$$

and

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=\left(\frac{lx_1}{a^2}+\frac{my_1}{b^2}-\frac{nz_1}{c^2}\right)^2. \qquad ...(5)$$

Now, (5), with the help of (4), reduces to

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0. \qquad (6)$$

Eliminating n from (4) and (5), we obtain

$$\frac{l^2}{a^4}(a^2z_1^2-c^2x_1^2)-\frac{2lmc^2x_1y_1}{a^2b^2}+\frac{m^2}{b^4}(b^2z_1^2-c^2y_1^2)=0.$$

If l_1, m_1, n_1, ; l_2, m_2, n_2 be the direction ratios of the two generators, this gives

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{b^2z_1^2-c^2y_1^2}{b^4}\cdot\frac{a^4}{a^2z_1^2-c^2x_1^2}$$

or

$$\frac{l_1l_2}{a^4(b^2z_1^2-c^2y_1^2)}=\frac{m_1m_2}{b^4(a^2z_1^2-c^2x_1^2)}=\frac{n_1n_2}{c^4(a^2y_1^2+c^2x_1^2)}.$$

Since

$$l_1l_2+m_1m_2+n_1n_2=0,$$

we obtain

$$a^4(b^2z_1^2-c^2y_1^2)+b^4(a^2z_1^2-c^2x_1^2)+c^4(a^2y_1^2+b^2x_1^2)=0,$$

or
$$b^2c^2x_1^2(c^2-b^2)+a^2c^2y_1^2(c^2-a^2)+a^2b^2z_1^2(a^2+b^2)=0,$$
or
$$(b^2-c^2)\frac{x_1^2}{a^2}+(a^2-c^2)\frac{y_1^2}{b^2}-(a^2+b^2)\frac{z_1^2}{c^2}=0.$$
We re-write it as
$$(a^2+b^2-c^2)\frac{x_1^2}{a^2}+(a^2+b^2-c^2)\frac{y_1^2}{b^2}-(a^2+b^2-c^2)\frac{z_1^2}{c^2}=x_1^2+y_1^2+z_1^2,$$
or
$$(a^2+b^2-c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}\right)=x_1^2+y_1^2+z_1^2.$$

Since now the point (x_1, y_1, z_1) lies on the hyperboloid, this reduces to
$$x_1^2+y_1^2+z_1^2=a^2+b^2-c^2.$$

Thus we see that the point of intersection of perpendicular generators lies on the curve of intersection of the hyperboloid and the director sphere
$$x^2+y^2+z^2=a^2+b^2-c^2.$$

Another method. Let PA, PB be two perpendicular generators through P and PC be the normal at P so that it is perpendicular to the tangent plane determined by PA and PB. The lines PA, PB, PC are mutually perpendicular and as such the three planes CPA, APB, BPC determined by them, taken in pairs, are also mutually perpendicular.

The plane CPA through the generator PA is the tangent plane at some point of PA and the plane CPB through the generator PB is the tangent plane at some point of PB. Also the plane APB is the tangent plane at P.

Thus the three planes CPA, APB and BPC are the mutually perpendicular tangent planes and as such their point of intersection P lies on the director sphere. Thus the locus of P is the curve of intersection of the hyperboloid with its director sphere.

Example

Show that the angle θ between the generators through any point P on a hyperboloid is given by
$$\cot\theta=p(r^2-a^2-b^2+c^2)/2abc,$$
where, p, is the perpendicular from the centre to the tangent plane at P and, r, is the distance of P from the centre. (*D.U., M.A. 1947, 59*)

The tangent plane at $P(x_1, y_1, z_1)$ is
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1. \qquad \ldots(1)$$

As in § 10·4, it can be shown that the direction ratios l, m, n, of the two generators through this point are given by the equations
$$\frac{lx_1}{a^2}+\frac{my_1}{b^2}-\frac{nz_1}{c^2}=0,$$
$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0.$$

Proceeding as in example 1, on page 169, we can show that angle θ between the lines is given by

$$\tan\theta=\frac{\sqrt{\left[-4\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)\left(-\frac{x_1^2}{a^4b^2c^2}-\frac{y_1^2}{b^4c^2a^2}+\frac{z_1^2}{c^4a^2b^2}\right)\right]}}{\frac{1}{a^2}\left(\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)+\frac{1}{b^2}\left(\frac{z_1^2}{c^4}+\frac{x_1^2}{a^4}\right)-\frac{1}{c^2}\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}\right)}$$

Now, p, the length of the perpendicular from the centre to the tangent plane (1) at (x_1, y_1, z_1) is given by

$$p=\frac{1}{\sqrt{\Sigma\frac{x_1^2}{a^4}}}\quad\text{or}\quad\frac{1}{p^2}=\Sigma\frac{x_1^2}{a^4}.$$

Also the denominator of the expression for $\tan\theta$

$$=\frac{1}{a^2b^2c^2}\left[\frac{x_1^2}{a^2}(c^2-b^2)+\frac{y_1^2}{b^2}(c^2-a^2)+\frac{z_1^2}{c^2}(a^2+b^2)\right]$$

$$=\frac{1}{a^2b^2c^2}\left[\frac{x_1^2}{a^2}(c^2-b^2-a^2)+\frac{y_1^2}{b^2}(c^2-a^2-b^2)+\frac{z_1^2}{c^2}(a^2+b^2-c^2)\right.$$
$$\left.+(x_1^2+y_1^2+z_1^2)\right]$$

$$=\frac{1}{a^2b^2c^2}\left[r^2-(a^2+b^2-c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}\right)\right]$$

$$=\frac{1}{a^2b^2c^2}\left[(r^2-a^2-b^2+c^2)\right].$$

$$\therefore\quad\tan\theta=\frac{\sqrt{\left[-\frac{4}{p^2}\left(-\frac{1}{a^2b^2c^2}\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}\right)\right]}}{(r^2-a^2-b^2+c^2)/a^2b^2c^2}$$
$$=2abc/p(r^2-a^2-b^2+c^2).$$

10·5. Central point. Line of Striction. Parameter of Distribution of a generator.

Def. 1. *The* **central point** *of any given generator, l, is the limiting position of its point of intersection with the line of shortest distance between it and another generator, m, of the same system; the limit being taken when, m, tends to coincide with l.*

With some sacrifice of precision, one may say that the central point of a given generator is the point of intersection of the generator and the shortest distance between it and a consecutive generator of the system.

Def. 2. *The locus of central points of generators is called* **line of striction.**

Def. 3. *The* **parameter of distribution** *of a generator, l, is $lim(\delta s/\delta\psi)$ where, δs, is the shortest distance and, $\delta\psi$, the angle between l and another generator m of the same system, the limit being taken when the generator m tends to coincide with the generator l.*

10·51. *To determine the central point of a generator.*

We consider generators of the system

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}.$$

Let any generator, l, of this system be

$$\frac{x-a\cos\varphi}{a\sin\varphi}=\frac{y-b\sin\varphi}{-b\cos\varphi}=\frac{z}{c}. \qquad \ldots(1)$$

We, now, consider any other generator, m,

$$\frac{x-a\cos\varphi'}{a\sin\varphi'}=\frac{y-b\sin\varphi'}{-b\cos\varphi'}=\frac{z}{c} \qquad \ldots(2)$$

of the same system.

Let the shortest distance between these generators meet them in P and Q respectively so that we have to find the limiting position of the point P on the generator l when $\varphi'\to\varphi$. Let C be the limit of P.

Since PQ is a chord of the hyperboloid, its limit will be a tangent line CD at the point C. Let l, m, n be the direction ratios of the shortest distance PQ and l_0, m_0, n_0 be those of its limit. We have

$$al\sin\varphi - bm\cos\varphi + cn = 0,$$
$$al\sin\varphi' - bm\cos\varphi' + cn = 0.$$

$$\therefore \quad \frac{al}{\cos\varphi'-\cos\varphi}=\frac{bm}{\sin\varphi'-\sin\varphi}=\frac{cn}{\sin(\varphi'-\varphi)},$$

or

$$\frac{al}{-\sin\tfrac{1}{2}(\varphi'+\phi)}=\frac{bm}{\cos\tfrac{1}{2}(\varphi'+\varphi)}=\frac{cn}{\cos\tfrac{1}{2}(\varphi'-\varphi)}.$$

Let $\varphi'\to\varphi$.

Thus we obtain

$$\frac{al_0}{-\sin\varphi}=\frac{bm_0}{\cos\varphi}=\frac{cn_0}{1}.$$

Let

$$[a(r\sin\varphi+\cos\varphi),\ b(\sin\varphi-r\cos\varphi),\ cr] \qquad \ldots(3)$$

be the central point C on the generator (1). The equation of the tangent plane at C is

$$\frac{x(r\sin\varphi+\cos\varphi)}{a}+\frac{y(\sin\varphi-r\cos\varphi)}{b}-\frac{zr}{c}=1.$$

Since the line CD with direction ratios l_0, m_0, n_0, lies on this tangent plane, we have

$$-\frac{\sin\varphi(r\sin\varphi+\cos\varphi)}{a^2}+\frac{\cos\varphi(\sin\varphi-r\cos\varphi)}{b^2}-\frac{r}{c^2}=0.$$

or

$$r\left[\frac{\sin^2\varphi}{a^2}+\frac{\cos^2\varphi}{b^2}+\frac{1}{c^2}\right]=\left[\frac{1}{b^2}-\frac{1}{a^2}\right]\sin\varphi\cos\varphi$$

or

$$r=\frac{c^2(a^2-b^2)\sin\varphi\cos\varphi}{(a^2b^2+a^2c^2\cos^2\varphi+b^2c^2\sin^2\varphi)},$$

so that we have obtained r.

Substituting this value of r in (3), we see that the co-ordinates of the central point $C(x, y, z)$ are given by

$$x=\frac{a^3(b^2+c^2)\cos\varphi}{k},\quad y=\frac{b^3(c^2+a^2)\sin\varphi}{k},\quad z=\frac{c^3(a^2-b^2)\sin\varphi\cos\varphi}{k}$$

where

$$k=a^2b^2+a^2c^2\cos^2\varphi+b^2c^2\sin^2\varphi.$$

Eliminating φ, we see that the *line of striction* is the curve of intersection of the hyperboloid with the cone

$$\frac{a^6(b^2+c^2)^2}{x^2}+\frac{b^6(c^2+a^2)^2}{y^2}-\frac{c^6(b^2-a^2)^2}{z^2}=0.$$

Ex. Find the central point for a generator of the second system and show that the line of striction is the same for either system.

10·52. *To determine the parameter of distribution of the generator, l.*

If $\delta\psi$ be the angle between the generators (1) and (2) of § 10·51, we have

$$\tan\delta\psi=\frac{\sqrt{[b^2c^2(\cos\varphi'-\cos\varphi)^2+c^2a^2(\sin\varphi'-\sin\varphi)^2+a^2b^2\sin^2(\varphi'-\varphi)]}}{a^2\sin\varphi\sin\varphi'+b^2\cos\varphi\cos\varphi'+c^2}$$

$$=2\sin\tfrac{1}{2}(\varphi'-\varphi)\frac{\sqrt{[b^2c^2\sin^2\tfrac{1}{2}(\varphi'+\varphi)+c^2a^2\cos^2\tfrac{1}{2}(\varphi'+\varphi)+a^2b^2\cos^2\tfrac{1}{2}(\varphi'-\varphi)]}}{a^2\sin\varphi\sin\varphi'+b^2\cos\varphi\cos\varphi'+c^2}$$

We write $\varphi'=\varphi+\delta\varphi$ so that $\delta\varphi\to0$ as $\varphi'\to\varphi$. Then, from above, we obtain

$$\frac{d\psi}{d\varphi}=\frac{\sqrt{[b^2c^2\sin^2\varphi+a^2c^2\cos^2\varphi+a^2b^2]}}{a^2\sin^2\varphi+b^2\cos^2\varphi+c^2}$$

Again we shall now find the *S.D.*, δs, between the two generators. Now the equation of the plane through (1) parallel to (2) is

$$\begin{vmatrix} x-a\cos\varphi, & y-b\sin\varphi, & z \\ a\sin\varphi, & -b\cos\varphi, & c \\ a\sin\varphi', & -b\cos\varphi' & c \end{vmatrix}=0$$

so that on cancelling a common factor $\sin\frac{1}{2}(\varphi'-\varphi)$, we obtain

$$-bcx\sin\tfrac{1}{2}(\varphi'+\varphi)+cay\cos\tfrac{1}{2}(\varphi'+\varphi)+abz\cos\tfrac{1}{2}(\varphi'-\varphi)$$
$$+abc\sin\tfrac{1}{2}(\varphi'-\varphi)=0.$$

The S.D., δs, which is the distance of the point

$$(a\cos\varphi',\ b\sin\varphi',\ 0)$$

from this plane is given by

$$\delta s = \frac{2abc \sin\frac{1}{2}(\varphi'-\varphi)}{\sqrt{[b^2c^2 \sin^2\frac{1}{2}(\varphi'+\varphi)+c^2a^2 \cos^2\frac{1}{2}(\varphi'+\varphi)+a^2b^2 \cos^2\frac{1}{2}(\varphi'-\varphi)]}}$$

Again putting $\varphi' = \varphi + \delta\varphi$, we obtain

$$\frac{ds}{d\varphi} = \frac{abc}{\sqrt{[b^2c^2 \sin^2\varphi + c^2a^2\cos^2\varphi + a^2b^2]}}.$$

$$\therefore \quad \frac{ds}{d\psi} = \frac{ds/d\varphi}{d\psi/d\varphi} = \frac{abc(a^2 \sin^2\varphi + b^2 \cos^2\varphi + c^2)}{b^2c^2 \sin^2\varphi + c^2a^2 \cos^2\varphi + a^2b^2}.$$

10·6. Hyperbolic paraboloid. We re-write the equation

$$x^2/a^2 - y^2/b^2 = 2z/c \qquad \ldots(1)$$

of a hyperbolic paraboloid in the form

$$\left[\frac{x}{a} - \frac{y}{b}\right]\left[\frac{x}{a} + \frac{y}{b}\right] = \frac{2z}{c},$$

which may again be re-written in either of the two forms

$$\frac{\frac{x}{a} - \frac{y}{b}}{\frac{z}{c}} = \frac{2}{\frac{x}{a} + \frac{y}{b}},$$

$$\frac{\frac{x}{a} - \frac{y}{b}}{2} = \frac{\frac{z}{c}}{\frac{x}{a} + \frac{y}{b}}.$$

Now, as in § 10·1 it can be shown that as λ and μ vary, each member of each of the system of lines

$$\frac{x}{a} - \frac{y}{b} = \frac{\lambda z}{c}, \quad 2 = \lambda\left[\frac{x}{a} + \frac{y}{b}\right] \qquad \ldots(A)$$

$$\frac{x}{a} - \frac{y}{b} = 2\mu, \quad \frac{z}{c} = \mu\left[\frac{x}{a} + \frac{y}{b}\right] \qquad \ldots(B)$$

lies wholly on the hyperbolic paraboloid (1).

Thus we see that a *hyperbolic paraboloid also admits of two systems of generating lines.*

As in the case of hyperboloid of one sheet, it can be shown that the following results hold good for the two systems of generating lines of a hyperbolic paraboloid also.

1. *Through every point of a hyperbolic paraboloid there passes one member of each system.*

2. *No two members of the same system intersect.*

3. *Any two generators belonging to different systems intersect and the plane through them is the tangent plane at their point of intersection.*

4. *The tangent plane at any point meets the paraboloid in two generators through the point.*

5. *The locus of the point of intersection of perpendicular generators is the curve of intersection of the paraboloid with the plane*

$$2cz+a^2-b^2=0.$$

An important Note. Since the generator

$$\frac{x}{a}-\frac{y}{b}=\frac{\lambda z}{c},\quad 2=\lambda\left(\frac{x}{a}+\frac{y}{b}\right),$$

lies in the plane

$$2=\lambda\left[\frac{x}{a}+\frac{y}{b}\right],$$

which is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}=0,$$

whatever value λ may have, we deduce that all the generators belonging to one system of the hyperbolic paraboloid

$$x^2/a^2-y^2/b^2=2z/c,$$

are parallel to the plane

$$x/a+y/b=0.$$

It may similarly be seen that the generators of the second system are also parallel to a plane, *viz.*,

$$x/a-y/b=0.$$

Example

Show that the polar lines with respect to the sphere

$$x^2+y^2+z^2=a^2$$

of the generators of the quadric

$$x^2-y^2=2az.$$

all lie on the quadric

$$x^2-y^2=-2az,$$

Re-writing the equation

$$x^2-y^2=2az,$$

as

$$(x-y)(x+y)=2az$$

we see that the two systems of generators of this quadric are

$$\left.\begin{matrix}x-y=2\lambda a\\ x+y=z/\lambda\end{matrix}\right\},\quad \left.\begin{matrix}x-y=2\mu z\\ x+y=a/\mu\end{matrix}\right\}.$$

Symmetric form of the λ generator is

$$\frac{x-\lambda a}{1}=\frac{y+\lambda a}{1}=\frac{z}{2\lambda}.$$

The polar plane of any point

$$(r+\lambda a,\ r-\lambda a,\ 2r\lambda),$$

on the λ generator *w.r.t.* the sphere

$$x^2+y^2+z^2=a^2$$

is

$$(r+\lambda a)x+(r-\lambda a)y+2r\lambda z=a^2,$$

i.e.,

$$r(x+y+2\lambda z)+a(\lambda x-hy-a)=0$$

so that the polar line of the λ generator is

$$x+y+2\lambda z=0,\ \lambda x-\lambda y-a=0.$$

Eliminating λ between these, we see that these polar lines lie on the quadric

$$x^2-y^2=-2az.$$

We may similarly treat the μ generators.

Exercises

1. Obtain equations for the two systems of generating lines on the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=4z,$$

and hence express the co-ordinates of a point on the surface as functions of two parameters. Find the direction cosines of the generators through $(\alpha, 0, \gamma)$ and show that the cosines of the angle between them is

$$(a^2-b^2+\gamma)/(a^2+b^2+\gamma).$$

2. Show that the projections of the generators of a hyperbolic paraboloid on any principal plane are tangents to the section by the plane.

3. Find the locus of the perpendiculars from the vertex of the paraboloid

$$x^2/a^2-y^2/b^2=2z/c$$

to the generators of one system.

[*Ans.* $x^2+y^2+2z^2\pm(a^2+b^2)xy/ab=0$.

10·7. Central point. Line of striction. Parameter of Distribution.

10·71. *To determine the central point of any generator of the system of generators*

$$\frac{x}{a}-\frac{y}{b}=\frac{\lambda z}{c},\ 2=\lambda\left(\frac{x}{a}+\frac{y}{b}\right).$$

Let any generator, l, of this system be

$$\frac{x}{a}-\frac{y}{b}=\frac{pz}{c},\ 2=p\left(\frac{x}{a}+\frac{y}{b}\right). \qquad \ldots(1)$$

We, now, consider any other generator, m, of the same system

$$\frac{x}{a}-\frac{y}{b}=\frac{p'z}{c},\ 2=p'\left(\frac{x}{a}+\frac{y}{b}\right). \qquad \ldots(2)$$

The direction ratios of these generators are

$$a,\ -b,\ 2c/p\ ;\ a,\ -b,\ 2c/p'.$$

If l, m, n be the direction ratios of the line of S.D. between (1) and (2), we have

$$al-bm+2cn/p=0,$$
$$al-bm+2cn/p'=0.$$

These give

$$1/a,\ 1/b,\ 0$$

as the direction ratios of the line of S.D. Being independent of p and p', we see that the line of S.D. is parallel to a fixed line.

Let (x_1, y_1, z_1) be the central point of (1). As in § 10·51, the limiting position of the line of S.D. is a line contained in the tangent plane

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{1}{c}(z+z_1)$$

at (x_1, y_1, z_1).

Thus we have

$$\frac{x_1}{a^3}-\frac{y_1}{b^3}=0. \qquad \ldots(3)$$

Also since (x_1, y_1, z_1) lies on (1), we have

$$\frac{x_1}{a}-\frac{y_1}{b}=\frac{pz_1}{c},\ 2=p\left(\frac{x_1}{a}+\frac{y_1}{b}\right) \qquad \ldots(4)$$

Solving (3) and (4), we obtain

$$x_1=\frac{2a^3}{p(a^2+b^2)},\ y_1=\frac{2b^3}{p(a^2+b^2)},\ z_1=\frac{2c(a^2-b^2)}{p(a^2+b^2)}.$$

Eliminating p, we see that the line of striction is the curve of intersection of the surface with the plane

$$x/a^3+y/b^3=0.$$

Ex. Find the central point of a generator of the second system and show that the corresponding line of striction is the curve of intersection of the surface with the plane

$$x/a^3+y/b^3=0.$$

10·72. *To determine the parameter of distribution.*

Let $\delta\psi$ and δs be the angle and the S.D. between (1) and (2).

We have

$$\tan\delta\psi=\frac{2c\sqrt{(a^2+b^2)}(p'-p)}{pp'(a^2+b^2)+4c^2}.$$

Let $p'=p+\delta p$ so that $\delta p\to 0$ as $p'\to p$.

$$\therefore\quad \frac{d\psi}{dp}=\frac{2c\sqrt{(a^2+b^2)}}{p^2(a^2+b^2)+4c^2}. \qquad \ldots(5)$$

Now the plane through the generator (1) and parallel to (2) is

$$\frac{x}{a}+\frac{y}{b}=\frac{2}{p}.$$

Also taking $z=0$, we see that $(a/p', b/p', 0)$ is a point on (2).

$$\therefore\quad \delta s=\frac{\frac{2}{p}-\frac{2}{p'}}{\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)}}=\frac{2(p'-p)ab}{pp'\sqrt{(a^2+b^2)}}. \qquad \ldots(6)$$

$\therefore$ as before

$$\frac{ds}{dp}=\frac{2ab}{p^2\sqrt{(a^2+b^2)}}.$$

$$\therefore\quad \frac{ds}{d\psi}=\frac{ab[p^2(a^2+b^2)+4c^2]}{cp^2(a^2+b^2)}$$

which is the parameter of distribution.

Ex. For the generator of the paraboloid

$$x^2/a^2-y^2/b^2=2z$$

given by

$$\frac{x}{a}-\frac{y}{b}=2\lambda, \quad \frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda},$$

prove that the parameter of distribution is

$$ab(a^2+b^2+4\lambda^2)/(a^2+b^2)$$

and the central point is

$$\left[\frac{2a^3\lambda}{a^2+b^2}, \frac{-2b^3\lambda}{a^2+b^2}, \frac{2(a^2-b^2)\lambda^2}{a^2+b^2}\right].$$

Prove that the central points of the systems of generators lie on the planes

$$x/a^3 \pm y/b^3=0. \qquad (D.U., M.A.)$$

10·8. General Consideration. We have seen that hyperboloid of one sheet and hyperbolic paraboloid each admit two systems of generators such that through each point of the surface there passes one member of each system and that two members of opposite systems intersect but no two members of the same system intersect. Also we know that through each point of a cone or a cylinder there passes one generator. Thus hyperboloids of one sheet, hyperbolic paraboloids, cones and cylinders are *ruled surfaces* inasmuch as they can be generated by straight lines.

We now proceed to examine the case of the general quadric in relation to the existence of generators.

10·81. Condition for a line to be a generator. *A straight line will be a generator of a quadric if three points of the line lie on the quadric.*

Let the quadric be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0. \quad ...(1)$$

The line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

will be a generator of the quadric, if the point

$$(lr+\alpha, mr+\beta, nr+\gamma)$$

on the line lies on the quadric for all values of r, *i.e.*, the equation obtained on substituting these co-ordinates for x, y, z in (1) is an identity. As this equation is a quadric in r, it will be an identity if it is satisfied for three values of r, *i.e.*, if three points of the line lie on the quadric.

Cor. 1. The quadric equation in r obtained above will be an identity if the co-efficients of r^2 and r and the constant term are separately zero. This gives

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0 \quad ...(2)$$

$$l(a\alpha+h\beta+g\gamma)+m(h\alpha+b\beta+f\gamma)+n(g\alpha+f\beta+c\gamma)=0 \quad ...(3)$$

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d=0 \quad ...(4)$$

The condition (4) simply means that the point (α, β, γ) lies on the quadric.

Since (2) is a homogeneous quadric equation and (3) is a homogeneous linear equation in l, m, n, these two equations will determine *two* sets of values of l, m, n. Thus we deduce that *through every point on a quadric there pass two lines, real, coincident or imaginary lying wholly on the quadric.*

Cor. 2. *A quadric can be drawn so as to contain* **three** *mutually skew lines as generators,* for the quadric determined by nine points, three on each line, will contain the three lines as generators.

10·9. Quadrics with real and distinct pairs of generating lines.

10·91. *Of all real central quadrics, hyperboloid of one sheet only possesses two real and distinct generators through a point.*

Let

$$ax^2+by^2+cz^2=1$$

be any central quadric.

The direction ratios l, m, n of any generator

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

of the quadric through the point (α, β, γ) are given by the equations

$$al^2+bm^2+cn^2=0,$$
$$al\alpha+bm\beta+cn\gamma=0.$$

Eliminating n from these, we obtain

$$a(a\alpha^2+c\gamma^2)l^2+2ab\alpha\beta lm+b(b\beta^2+c\gamma^2)m^2=0.$$

Its roots will be real and distinct if, and only if,

$$4a^2b^2\alpha^2\beta^2-4ab(a\alpha^2+c\gamma^2)(b\beta^2+c\gamma^2)>0,$$

i.e., if

$$-4abc\gamma^2(a\alpha^2+b\beta^2+c\gamma^2)>0.$$

Since $a\alpha^2+b\beta^2+c\gamma^2=1$, we see that the roots will be real and distinct, if and only if,

abc is negative.

Now this will be the case if a, b, c are all negative or one negative and two positive. In the former case the quadric itself is imaginary and in the latter it is a hyperboloid of one sheet.

10·92. *Of the two paraboloids, hyperbolic paraboloid only possesses two real and distinct generators through a point.*

In the case of the paraboloid

$$ax^2+by^2=2cz,$$

the direction ratios, l, m, n of the generating lines through a point (α, β, γ) of the surface are given by

$$al^2+bm^2=0, \qquad \ldots(1)$$
$$al\alpha+bm\beta-nc=0. \qquad \ldots(2)$$

The equation (1) shows that for real values of l and m, we must have a and b with opposite signs, *i.e.*, the paraboloid must be hyperbolic.

10·10. Lines intersecting three lines. *An infinite number of lines can be drawn meeting three given mutually skew lines.* For the quadric through the three given mutually skew lines a, b, c, the three lines will be generators of one system and all the other generators of the other system will intersect a, b, and c.

In fact the quadric through three given mutually skew lines can be determined as the locus of lines which intersect the three given lines.

Thus the locus determined in § 4·41 on page 65 is really the equation of the quadric through the three lines

$$u_r=0=v_r \; ; \; r=1, 2, 3.$$

Exercises

1. Find the equation of the quadric containing the three lines

$$y=b, \; z=-c \; ; \; z=c, \; x=-a \; ; \; x=a, \; y=-b.$$

Also obtain the equations of its two systems of generators. (See Ex. 2. page 65).

[*Ans.* $ayz+bzx+cxy+abc=0$; $y-b+\lambda(z+c)=0$, $b(x+a)-\lambda(az+cx)=0$.
$\mu(az+cx)-(z+c)=0$, $\mu b(x+a)+(y-b)=0$.

2. Find the equations of the hyperboloid through the three lines

$$y-z=1, \; x=0 \; ; \; z-x=1, \; y=0 \; ; \; x-y=1, \; z=0.$$

Also obtain the equations of its two systems of generators.

[*Ans.* $x^2+y^2+z^2-2xy-2yz-2zx=1$; $x-y-1=\lambda z$, $\lambda(x-y+1)=2x+2y-z$,
$x-y-1=\lambda(2x+2y-z)$, $\lambda(x-y+1)=z$.

3. The generators of one system of a hyperbolic paraboloid are parallel to the plane $lx+my+nz=0$ and the lines

$$ax+by=0=z+c \; ; \; ax-by=0=z-c$$

are two members of the same system.

Show that the equation of the paraboloid is

$$abz(lx+my+nz)=c(a^2mx+b^2ly+abcn).$$

(See Ex. 3, p. 65)

4. Show that two straight lines can be drawn intersecting four given mutually skew lines.

Examples

1. *From a fixed point $A(f, g, h)$ perpendiculars are let fall on three conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \; ;$$

prove that the plane passing through the feet of the perpendiculars goes through the fixed point

$$\left[\frac{a^2f}{a^2+b^2+c^2}, \; \frac{b^2g}{a^2+b^2+c^2}, \; \frac{c^2h}{a^2+b^2+c^2}\right]. \qquad (B.U.)$$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of their conjugate semi-diameters.

Equations of OP are

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1},$$

so that (rx_1, ry_1, rz_1) are the general co-ordinates of any point on this

line. The line joining (f, g, h) and (rx_1, ry_1, rz_1) will be perpendicular to OP, if

$$(rx_1-f)x_1+(ry_1-g)y_1+(rz_1-h)z_1=0,$$

i.e., if,

$$r=\frac{fx_1+gy_1+hz_1}{x_1^2+y_1^2+z_1^2}.$$

Therefore, the foot of the perpendicular L from (f, g, h) to OP is

$$\left(\frac{\Sigma fx_1}{\Sigma x_1^2}x_1, \frac{\Sigma fx_1}{\Sigma x_1^2}y_1, \frac{\Sigma fx_1}{\Sigma x_1^2}z_1\right).$$

Similarly, the feet M, N of the perpendiculars to OQ, OR are

$$\left(\frac{\Sigma fx_2}{\Sigma x_2^2}x_2, \frac{\Sigma fx_2}{\Sigma x_2^2}y_2, \frac{\Sigma fx_2}{\Sigma x_2^2}z_2\right),$$

and

$$\left(\frac{\Sigma fx_3}{\Sigma x_3^2}x_3, \frac{\Sigma fx_3}{\Sigma x_3^2}y_3, \frac{\Sigma fx_3}{\Sigma x_3^2}z_3\right),$$

The plane LMN is

$$\begin{vmatrix} x, & y, & z, & 1 \\ \frac{\Sigma fx_1}{\Sigma x_1^2}x_1, & \frac{\Sigma fx_1}{\Sigma x_1^2}y_1, & \frac{\Sigma fx_1}{\Sigma x_1^2}z_1, & 1 \\ \frac{\Sigma fx_2}{\Sigma x_2^2}x_2, & \frac{\Sigma fx_2}{\Sigma x_2^2}y_2, & \frac{\Sigma fx_2}{\Sigma x_2^2}z_2, & 1 \\ \frac{\Sigma fx_3}{\Sigma x_3^2}x_3, & \frac{\Sigma fx_3}{\Sigma x_3^2}y_3, & \frac{\Sigma fx_3}{\Sigma x_3^2}z_3, & 1 \end{vmatrix}=0,$$

or

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1\Sigma fx_1, & y_1\Sigma fx_1, & z_1\Sigma fx_1, & \Sigma x_1^2 \\ x_2\Sigma fx_2, & y_2\Sigma fx_2, & z_2\Sigma fx_2, & \Sigma x_2^2 \\ x_3\Sigma fx_3, & y_3\Sigma fx_3, & z_3\Sigma fx_3, & \Sigma x_3^2 \end{vmatrix}=0.$$

Adding third and fourth rows to the second and making use of the relation in § 8·8, this becomes

$$\begin{vmatrix} x, & y, & z, & 1 \\ a^2f, & b^2g, & c^2h, & \Sigma a^2 \\ x_2\Sigma fx_2, & y_2\Sigma fx_2, & z_2\Sigma fx_2, & \Sigma x_2^2 \\ x_3\Sigma fx_3, & y_3\Sigma fx_3, & z_3\Sigma fx_3, & \Sigma x_3^2 \end{vmatrix}=0,$$

or
$$\begin{vmatrix} x, & y, & z, & 1 \\ a^2f/\Sigma a^2, & b^2g/\Sigma a^2, & c^2h/\Sigma a^2, & 1 \\ x_2\Sigma fx_2, & y_2\Sigma fx_2, & z_2\Sigma fx_2, & \Sigma x_2^2 \\ x_3\Sigma fx_3, & y_3\Sigma fx_3, & z_3\Sigma fx_3, & \Sigma x_3^2 \end{vmatrix} = 0,$$

This form of the equation of the plane clearly shows that it passes through the point
$$\left(\frac{a^2f}{\Sigma a^2}, \frac{b^2g}{\Sigma a^2}, \frac{c^2h}{\Sigma a^2}\right).$$

2. *Show that the normals to the ellipsoid*
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$
at all points of a central circular section are parallel to a fixed plane. Find the angle which this plane makes with the plane of the section.

Consider the central circular section
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{\sqrt{(a^2-b^2)}}{a}x+\frac{\sqrt{(b^2-c^2)}}{c}z=0.$$

The direction ratios of the normal to the ellipsoid at any point (f, g, h) of the section are
$$f/a^2, \ g/b^2, \ h/c^2.$$

Also we have the relation
$$\frac{\sqrt{(a^2-b^2)}}{a}f+\frac{\sqrt{(b^2-c^2)}}{c}h=0,$$
which we re-write as
$$a\sqrt{(a^2-b^2)}\,.\,\frac{f}{a^2}+0\,.\,\frac{g}{b^2}+c\sqrt{(b^2-c^2)}\,.\,\frac{h}{c^2}=0.$$

This relation shows that the normals are parallel to the fixed plane
$$a\sqrt{(a^2-b^2)}x+0y+c\sqrt{(b^2-c^2)}z=0.$$

If, θ, be the angle between this plane and the plane of the section, we have
$$\cos\theta=\frac{(a^2-b^2)+(b^2-c^2)}{\sqrt{\left(\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right)}\sqrt{\left[a^2(a^2-b^2)+c^2(b^2-c^2)\right]}}$$
$$=\frac{a^2-c^2}{\sqrt{\left[b^2\left(\frac{1}{c^2}-\frac{1}{a^2}\right)(a^2-c^2)(a^2+c^2-b^2)\right]}}$$
$$=\frac{ac}{b\sqrt{(a^2+c^2-b^2)}}.$$
$$\therefore \quad \theta=\cos^{-1}[ac/b\sqrt{(a^2+c^2-b^2)}].$$

3. *The generators through a point P on the hyperboloid*

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

meet the principal plane $z=0$ in points A and B such that the median of the triangle PAB through P is parallel to the fixed plane

$$lx+my+nz=0\ ;$$

show that P lies on the curve of intersection of the hyperboloid with the surface

$$z(lx+my)+n(c^2+z^2)=0.$$

The equations of the line AB where the tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1,$$

at P (x_1, y_1, z_1) of the hyperboloid, containing as it does the generators through P, meets the plane $x=0$ are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0. \qquad \ldots(1)$$

Let $(f, g, 0)$ be the mid-point of AB. The equations of the chord of the principal elliptical section

$$x^2/a^2+y^2/b^2=1,\ z=0,$$

with $(f, g, 0)$ as its middle point is

$$\frac{fx}{a^2}+\frac{gy}{b^2}=\frac{f^2}{a^2}+\frac{g^2}{b^2},\ z=0 \qquad \ldots(2)$$

Comparing (1) and (2), we have

$$x_1=\frac{f}{f^2/a^2+g^2/b^2},\ y_1=\frac{g}{f^2/a^2+g^2/b^2}.$$

These give

$$f=\frac{x_1}{x_1^2/a^2+y_1^2/b^2}=\frac{x_1}{1+z_1^2/c^2} \qquad \ldots(3)$$

$$g=\frac{y_1}{x_1^2/a^2+y_1^2/b^2}=\frac{y_1}{1+z_1^2/c^2}. \qquad \ldots(4)$$

Also the median of the $\triangle PAB$ through P being parallel to

$$lx+my+nz=0,$$

we have

$$l(x_1-f)+m(y_1-g)+nz_1=0. \qquad \ldots(5)$$

Eliminating f, g from (3), (4) and (5) we obtain

$$z_1(lx_1+my_1)+n(c^2+z_1^2)=0.$$

Thus we have the result as required.

Revision Exercises III

1. Prove that if θ is the angle between the central radius to the point $P(x, y, z)$ on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and normal at P,

$$\tan^2\theta=\Sigma y^2z^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right).$$

2. Prove that the common tangents of the three ellipsoids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad \frac{x^2}{b^2}+\frac{y^2}{c^2}+\frac{z^2}{a^2}=1,\quad \frac{x^2}{c^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}=1,$$

touch a sphere of radius

$$\sqrt{\left(\frac{a^2+b^2+c^2}{3}\right)} \qquad (B.U.\ 1956)$$

and that the points of contact of the planes lie on a sphere of radius

$$\sqrt{\left(\frac{a^4+b^4+c^4}{a^2+b^2+c^2}\right)}. \qquad (M.T.)$$

3. Show that if three central radii of an ellipsoid be mutually perpendicular, the plane passing through their extremities will envelope a sphere.

4. Prove that six normals can be drawn from any point P to a central quadric surface and that those six normals are generators of a quadric cone with vertex at P.

Prove that the conic in which the cone meets any one of the principal planes of the quadric surface remains fixed when P moves along a straight line perpendicular to that plane. *(Birmingham)*

5. Show that the length of the normal chord at any point (x, y, z) of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

is $2q^4/4p^2$ where

$$\frac{1}{p^2}=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4},$$

$$\frac{1}{q^4}=\frac{x^2}{a^6}+\frac{y^2}{b^6}+\frac{z^2}{c^6}. \qquad (M.T.)$$

6. If p_1, p_2, p_3, and π_1, π_2, π_3 be the perpendiculars from the extremities P_1, P_2, P_3, of conjugate semi-diameters on the two central circular sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then

$$p_1\pi_1+p_2\pi_2+p_3\pi_3=\frac{a^2c^2(a^2+c^2+2b^2)}{a^2-c^2}.$$

7. Show that the locus of points on the quadric $ax^2+by^2+cz^2=1$, the normals at which intersect the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is the curve of intersection with the quadric

$$l(b-c)yz+m(c-a)zx+n(a-b)xy-(n\beta-m\gamma)ax-(l\gamma-n\alpha)by-(m\alpha-l\beta)cz=0.$$

8. If $r_1, r_2, r_3, r_4, r_5, r_6$, are the lengths of the normals drawn from any point to a central conicoid and $p_1, p_2, p_3, p_4, p_5, p_6$, are the lengths of the perpendiculars from its centre to the tangent planes at their feet,

$$p_1r_1+p_2r_2+p_3r_3+p_4r_4+p_5p_5+p_6r_6$$

is constant.

9. Two planes are drawn through the six feet of the normals drawn to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ from a given point (f, g, h); each plane containing three; prove that if (α, β, γ), $(\alpha', \beta', \gamma')$ be the poles of these planes with respect to the ellipsoid then,

$$\alpha\alpha'+a^2=\beta\beta'+b^2=\gamma\gamma'+c^2$$

and

$$f(\alpha+\alpha')+g(\beta+\beta')+h(\gamma+\gamma')=0.$$

10. If three of the feet of the normals from a point to the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

lie on the plane

$$lx+my+nz=p,$$

show that the equation of the plane through the other three is

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

Also, show that if one of the planes contains the extremities of three conjugate semi-diameters, the other plane cuts the co-ordinate planes in triangle whose centroid lies on a coaxal ellipsoid. *(B.U. 1929)*

11. Pairs of planes are drawn which are conjugate with respect to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the first member of each pair passing through the line

$$y=mx,\ z=k$$

and the second member of each pair passing through the line

$$y=-mx,\ z=-k\ ;$$

prove that the line of intersection of the two members of any pair lie on the surface $(b^2-a^2m^2)\ (z^2-k^2)+(y^2-m^2x^2)\ (c^2+k^2)=0$. *(Bm. U. 1926)*

12. The normal to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ at a point P meets the plane $z=0$ at G and GO is drawn perpendicular to this plane and equal to GP. Show that the locus of Q is the surface

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1\ ;$$

show also that if, N is the foot of the perpendicular from P to the principal plane on which G lies and the normal at Q to its locus meets this plane at K, then G is the midpoint of KN. *(L.U. 1914)*

13. Through a given point (α, β, γ) planes are drawn parallel to three conjugate diametral planes of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. Show that the sum of the ratios of the areas of the sections by these planes to the areas of the parallel central planes is

$$3-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}.$$

14. If A_1, A_2, A_3, are the areas of the sections of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

by the diametral planes of three mutually perpendicular semi-diameters of lengths r_1, r_2, r_3, show that

$$\frac{A_1^2}{r_1^2}+\frac{A_2^2}{r_2^2}+\frac{A_3^2}{r_3^2}=\pi^2\left(\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}\right).$$

15. If through a given point (f, g, h) lines be drawn each of which is an axis of some plane section of

$$ax^2+by^2+cz^2=1,$$

such lines describe the cone

$$\frac{af\ (b-c)}{x-f}+\frac{bg\ (c-a)}{y-g}+\frac{ch\ (a-b)}{z-h}=0.$$

16. If a plane $lx+my+nz=p$ cuts the surface $ax^2+by^2+cz^2=1$ in a parabolic section, prove that the direction cosines of its axis are proportional to l/a, m/b, n/c and the co-ordinates of the vertex of the parabola satisfy the equation

$$\frac{ax}{l}\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{by}{m}\left(\frac{1}{c}-\frac{1}{a}\right)+\frac{cz}{n}\left(\frac{1}{a}-\frac{1}{b}\right)=0.$$

(A.U. 1920)

17. Prove that the generating lines through any point P on the section $z=c$ of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ meet the principal section by the plane $z=0$ at the ends of a pair of conjugate diameters. *(M.T.)*

18. The generators of opposite systems drawn through the extremities A, B of semi-conjugate diameters of the principal elliptic section of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ meet in P; show that the median through P of the triangle PAB lies on the cone

$$\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=\left(\frac{z}{c}\pm 1\right)^2.$$

19. Prove that tangent planes to $x^2/a^2+y^2/b^2-z^2/c^2=1$ which are parallel to the tangent planes to

$$\frac{x^2}{\frac{1}{b^2}-\frac{1}{c^2}}+\frac{y^2}{\frac{1}{a^2}-\frac{1}{c^2}}+\frac{z^2}{\frac{1}{a^2}-\frac{1}{b^2}}=0$$

meet the surface in perpendicular generators. *(P.U., M.A. 1938)*

20. Show that the generators of the surface $x^2+y^2-z^2=1$ which intersect on XOY plane are at right angles.

21. Show that the points on the quadric

$$ax^2+by^2+cz^2+d=0,$$

at which the generators are perpendicular lie on the cylinder

$$(c-a)x^2+(c-b)y^2+cd(a+b)/ab=0. \qquad (M.T.)$$

22. If $(a\cos\theta\sec\varphi,\ b\sin\theta\sec\varphi,\ c\tan\varphi)$ is a point on the generating line

$$\frac{x}{a}+\frac{z}{c}=\lambda\left\{1+\frac{y}{b}\right\},\quad \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left\{1-\frac{y}{b}\right\}$$

of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$, prove that for points along the generator, $\theta-\varphi$ is constant. *(P.U., M.A. 1943)*

23. Show that the most general quadric surface which has the lines

$$x=0,\ y=0\ ;\ x=0,\ z=c\ ;\ y=0,\ z=-c$$

as generators is

$$fy(z-c)+gx(z+c)+hxy=0,$$

where f, g, h are arbitrary constants. *(M.T.)*

24. Find equations in symmetrical form for the line of intersection of the two planes whose equations are

$$x+y=2(\lambda+\mu)(z-1),\ (x-y)=2(\lambda-\mu)(z-1),$$

where λ and μ are constants. Find also the co-ordinates of the point in which this line meets the plane $z=0$.

If now λ and μ are taken to be variable parameters connected by the relation $\lambda^2+\mu^2=1$, show that line traces out a right circular cone.

CHAPTER XI

GENERAL EQUATION OF THE SECOND DEGREE

Reduction to canonical forms and Classification of Quadrics

11·1. A quadric has been defined as the locus of a point satisfying an equation of the second degree. Thus a quadric is the locus of a point satisfying an equation of the type

$$F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

which we may re-write as

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d=0, \qquad \ldots(1)$$

splitting the set of all terms into three homogeneous sub-sets.

We have considered so far special forms of the equations of the second degree in order to discuss geometrical properties of the various types of quadrics. In this chapter we shall see how the general equation of a second degree can be reduced to simpler forms and also thus classify the types of quadrics.

Firstly we shall proceed to determine the equations of various loci associated with a quadric given by a general second degree equation. In this connection we shall start obtaining a quadric in r, which will play a very important role in connection with the determination of the equations of these loci.

Consider any point (α, β, γ) and any line through the same with direction cosines (l, m, n). The co-ordinates of the point on this line at a distance r from (α, β, γ) are

$$(lr+\alpha, mr+\beta, nr+\gamma).$$

This point will lie on the quadric

$$F(x, y, z) \equiv \Sigma(ax^2+2fyz)+2\Sigma ux+d=0,$$

for values of r satisfying the equation

$$\Sigma[a(lr+\alpha)^2+2f(mr+\beta)(nr+\gamma)]+2\Sigma u(lr+\alpha)+d=0.$$

i.e.,

$$r^2\Sigma(al^2+2fmn)+2r[l(a\alpha+h\beta+g\gamma+u)+m(h\alpha+b\beta+f\gamma+v)+n(g\alpha+f\beta+c\gamma+w)]+F(\alpha, \beta, \gamma)=0 \qquad \ldots(2)$$

which is a quadric, in r. Thus if r_1, r_2 be the roots of this quadric, the two points of intersection of the line with the quadric are

$$(lr_1+\alpha, mr_1+\beta, nr_1+\gamma), (lr_2+\alpha, mr_2+\beta, nr_2+\gamma).$$

Note. It may be noted that the equation (2) can be written as

$$r^2\Sigma(al^2+2fmn)+r\left(l\frac{\partial F}{\partial \alpha}+m\frac{\partial F}{\partial \beta}+n\frac{\partial F}{\partial \gamma}\right)+F(\alpha, \beta, \gamma)=0$$

where $\partial F/\partial\alpha$, $\partial F/\partial\beta$. $\partial F/\partial\gamma$ denote the values of the partial derivative of F *w. r.* to x, y, z respectively at the point (α, β, γ).

11·11. Tangent plane at (α, β, γ). Suppose that the point (α, β, γ) lies on the quadric so that

$$F(\alpha, \beta, \gamma)=0$$

and accordingly one root of the quadric equation (2) is zero. The vanishing of one value of r is also a simple consequence of the fact that one of the two points of intersection of the quadric with every line through a point of the quadric coincides with the point in question.

A line through (α, β, γ) with direction cosines l, m, n will be a tangent line if the second point of intersection also coincides with (α, β, γ) *i.e.*, if the second value of r, as given by (2), is also zero. This will be so if the co-efficient of r is also zero, *i.e.*,

$$l(a\alpha+h\beta+g\gamma+u)+m(h\alpha+b\beta+f\gamma+v)+n(g\alpha+f\beta+c\gamma+w)=0 \quad \ldots(3)$$

which is thus the condition for the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(4)$$

to be a tangent line at (α, β, γ). The locus of the tangent lines through (α, β, γ), obtained on eliminating l, m, n between (3) and (4) is,

$$\Sigma(x-a)(a\alpha+h\beta+g\gamma+u)=0,$$

i.e.,
$$\Sigma x(a\alpha+h\beta+g\gamma+u)=\Sigma\alpha(a\alpha+h\beta+g\gamma+u).$$

Adding $u\alpha+v\beta+w\gamma+d$ to both sides, we get

$$\Sigma x(a\alpha+h\beta+g\gamma+u)+(u\alpha+v\beta+w\gamma+d)=F(\alpha, \beta, \gamma)=0.$$

Thus the locus of the tangent lines at (α, β, γ) is

$$\Sigma x(a\alpha+h\beta+g\gamma+u)+(u\alpha+v\beta+w\gamma+d)=0,$$

which is a plane called the *tangent plane* at (α, β, γ).

11·12. Normal at (α, β, γ). The line through (α, β, γ), perpendicular to the tangent plane thereat, *viz.*,

$$\frac{x-\alpha}{a\alpha+h\beta+g\gamma+u}=\frac{y-\beta}{h\alpha+b\beta+f\gamma+v}=\frac{z-\gamma}{g\alpha+f\beta+c\gamma+w},$$

is the normal at (α, β, γ).

11·13. Enveloping cone from a point. Suppose now that (α, β, γ) is any point necessarily on the quadric. Then any line through (α, β, γ) with direction lines (l, m, n) will touch the quadric *i.e.*, meet the same in two coincident points, if the two roots of the quadric equation in r, are equal. The condition for this is

$$[\Sigma l(a\alpha+h\beta+g\gamma+u)]^2=[\Sigma(al^2+2fmn)]\,F(\alpha, \beta, \gamma) \quad \ldots(5)$$

The locus of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(6)$$

through (α, β, γ) touching the quadric, obtained on eliminating l, m, n, between (5) and (6) is

$$[\Sigma(x-\alpha)(a\alpha+h\beta+g\gamma+u)]^2$$
$$=[\Sigma\alpha(x-\alpha)^2+2f(y-\beta)(z-\gamma)]F(\alpha, \beta, \gamma) \quad \ldots(7)$$

To put this equation in a convenient form, we write

$$S \equiv F(x, y, z), \qquad S_1 \equiv F(\alpha, \beta, \gamma),$$
$$T \equiv \Sigma x(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + w\gamma + d).$$

Then (7) can be re-written as

$$(T - S_1)^2 = S_1(S + S_1 - 2T)$$

i.e.,

$$SS_1 = T^2$$

which is the equation of the enveloping cone of the quadric $S=0$ with the point (α, β, γ) as its vertex.

11·14. Enveloping Cylinder. Suppose now that (l, m, n) are given and we require the locus of tangent lines with direction cosines (l, m, n). If (α, β, γ) be any point on any such tangent line, we have the condition

$$[\Sigma l(a\alpha + h\beta + g\gamma + u)]^2 = [\Sigma(al^2 + 2fmn)]F(\alpha, \beta, \gamma)$$

as obtained in § 11·13 above. Thus the required locus is

$$[\Sigma l(a\alpha + hy + gz + u)]^2 = \Sigma(al^2 + 2fmn)\, F(x, y, z),$$

known as Enveloping Cylinder.

11·15. Section with a given centre. Suppose now that (α, β, γ) is a given point. Then any chord with direction cosines l, m, n through (α, β, γ) will be bisected thereat if the sum of the two roots of the r-quadratic (2) is zero, *i.e.*,

$$\Sigma l(a\alpha + h\beta + g\gamma + u) = 0, \qquad \ldots(8)$$

so that the locus of the chord

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \ldots(9)$$

through (α, β, γ) and bisected thereat, obtained on eliminating l, m, n from (8) and (9) is

$$\Sigma(x-\alpha)(a\alpha + h\beta + g\gamma + u) = 0,$$

which, we may re-write as,

$$T = S_1.$$

Clearly the plane $T=S_1$, meets the quadric in a conic with its centre at (α, β, γ).

11·16. Polar plane of a point. If any line through $A(\alpha, \beta, \gamma)$ meet the quadric in Q, R and a point P is taken on the line such that the points A and P divide QR, internally and externally in the same ratio, then the locus of P for different lines through A is a plane called the polar plane of A with respect to the quadric. It is easily seen that if A and P divide QR internally and externally in the same ratio then the points Q and R also divide AP internally and externally in the same ratio.

Consider any line through $A(\alpha, \beta, \gamma)$ and let P be the point (x, y, z). The point dividing AP in the ratio $\lambda : 1$ is

$$\left(\frac{\lambda x + \alpha}{\lambda + 1}, \frac{\lambda y + \beta}{\lambda + 1}, \frac{\lambda z + \gamma}{\lambda + 1}\right).$$

It will lie on the quadric

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d=0,$$

$$\text{if } \Sigma\left[\, a\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2+2f\left(\frac{\lambda y+\beta}{\lambda+1}\right)\left(\frac{\lambda z+\gamma}{\lambda+1}\right)\right]+2\Sigma u\left(\frac{\lambda x+\alpha}{\lambda+1}\right)+d=0.$$

i.e., $\lambda^2F(x, y, z)+2\lambda[x(a\alpha+h\beta+g\gamma+u)+y(h\alpha+b\beta+f\gamma+v)$
$+z(g\alpha+f\beta+c\gamma+w)+(u\alpha+v\beta+w\gamma+d)]+F(\alpha, \beta, \gamma)=0.$

The two values of λ give the two ratios in which the points Q and R divide AP. In order that Q and R may divide AP internally and externally in the same ratio, the sum of the two values of λ should be zero, *i.e.*,

$$x(a\alpha+h\beta+g\gamma+u)+y(h\alpha+b\beta+f\gamma+v)+z(g\alpha+f\beta+c\gamma+w)+$$
$$(u\alpha+v\beta+w\gamma+d)=0, \qquad \ldots(10)$$

which is the required locus of the point $P(x, y, z)$.

Thus (10) is the required equation of the polar plane.

Note. The notions of conjugate points, conjugate planes, conjugate lines and polar lines can be introduced as in the case of particular forms of equations in the preceding chapters.

11·21. Some preliminaries to reduction and classification. In this section we shall state some points which will prove useful in the problem of reduction and classification.

In the following discussions, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

to be denoted by D will play an important part.

We may verify that

$$D=abc+2fgh-af^2-bg^2-ch^2.$$

As before, A, B, C, F, G, H will denote the co-factors of a, b, c, f, g, h respectively in D so that we have

$$A=bc-f^2,\ B=ca-g^2,\ C=ab-h^2;$$
$$F=gh-af,\ G=hf-bg,\ H=fg-ch.$$

It can be easily verified that

$$\left.\begin{aligned} &BC-F^2=aD,\ CA-G^2=bD,\ AB-H^2=cD;\\ &GH-AF=fD,\ HF-BG=gD,\ FG-CH=hD. \end{aligned}\right\} \qquad \ldots(1)$$

Also we have

$$aA+hH+gG=D,\ hA+bH+fG=0,\ gA+fH+cG=0;$$
$$aH+hB+gF=0,\ hH+bB+fF=D,\ gH+fB+cF=0;$$
$$aG+hF+gC=0,\ hG+bF+fC=0,\ gG+fF+cC=D.$$

11·22. *If $D=0$, then we have*

$$BC=F^2,\ CA=G^2,\ AB=H^2.$$
$$GH=AF,\ HF=BG,\ FG=CH.$$

These follow from the relation (1) in § 11·21 above.

Ex. If $D=0$ and $A=0$, prove that $H=0$, $G=0$. Also prove that if $D=0$ and $H=0$ then

either $A=0, H=0, G=0$ or $H=0, B=0, F=0$.

Further prove that if $D=0$, $A=0$, $B=0$, then F, G, H must all be zero but C may or may not be zero.

11·23. *If* $D=0$ *and* $A+B+C=0$, *then*

$$A, B, C, F, G, H$$

are all zero.

As $D=0$, we have

$$BC=F^2,\ CA=G^2,\ AB=H^2,$$

so that A, B, C are all of the same sign. Since, also $A+B+C=0$, we deduce that

$$A=0,\ B=0,\ C=0.$$

Also then

$$F^2=BC=0 \text{ so that } F=0.$$

Similarly $G=0$, $H=0$.

11·24. The two planes

$$p_1x+q_1y+r_1z+s_1=0$$
$$p_2x+q_2y+r_2z+s_2=0$$

will be

(*i*) same if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}=0, \quad \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}=0, \quad \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix}=0,$$

(*ii*) parallel but not same if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}=0, \quad \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}=0, \quad \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix}\neq 0.$$

(*iii*) neither parallel nor same, *i.e.*, will intersect in a straight line if

$$\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}\neq 0 \text{ or } \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}\neq 0.$$

11·25. Three homogeneous linear equations

$$a_1x+b_1y+c_1z=0,$$
$$a_2x+b_2y+c_2z=0,$$
$$a_3x+b_3y+c_3z=0,$$

will possess a non-zero solution, *i.e.*, a solution wherefor x, y, z are not all zero, if and only if,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

11·3. Diametral plane conjugate to a given direction. We know [Refer equation (2) p. 223] that if l, m, n, be the direction cosines of any chord and (x, y, z) the midpoint of the same, then we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0. \qquad \ldots(1)$$

Thus if l, m, n be supposed given, then the equation of the locus of the midpoints (x, y, z) of parallel chords with direction cosines l, m, n is given by (1) above. This locus is a plane called *the diametral plane conjugate to the direction* l, m, n. We can re-write the equation (1) of the diametral plane conjugate to l, m, n as

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)+(ul+vm+wn)=0. \qquad \ldots(2)$$

Note. In this connection we should remember that there does not necessarily correspond a diametral plane conjugate to *every* given direction. Thus we see from above that there is no diametral plane conjugate to the direction l, m, n if these are such that the co-efficient of x, y, z in the equation (2) are all zero,

i.e.,

$$al+hm+gn=0,$$
$$hl+bm+fn=0,$$
$$gl+fm+cn=0.$$

As l, m, n are not all zero, this can of course happen only if

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

11·4. Principal Directions and Principal Planes. A direction l, m, n is said to be *principal*, if it is perpendicular to the diametral plane conjugate to the same. Also then the corresponding conjugate diametral plane is called a *principal plane.*

Thus, l, m, n will be a principal direction if and only if the direction ratios

$$al+hm+gn,\ hl+bm+fn,\ gl+fm+cn$$

of the normal to the corresponding conjugate diametral plane are proportional to

$$l, m, n,$$

i.e., there exists a number λ such that

$$al+hm+gn=l\lambda,$$
$$hl+bm+fn=m\lambda,$$
$$gl+fm+cn=n\lambda.$$

We re-write these as

$$(a-\lambda)l+hm+gn=0, \quad \ldots(1)$$
$$hl+(b-\lambda)m+fn=0, \quad \ldots(2)$$
$$gl+fm+(c-\lambda)n=0. \quad \ldots(3)$$

These three linear homogeneous equations in l, m, n will possess a non-zero solution in l, m, n if and only if,

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0.$$

On expanding this determinant, we see that λ must be a root of the cubic

$$\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0. \quad \ldots(4)$$

This cubic is known as the **Discriminating cubic** and each root of the same is called a **characteristic root.**

The equation (4) has three roots which may not all be distinct. Also to each root of (4) corresponds at least one principal direction l, m, n obtained on solving any two of the equations (1), (2) and (3).

Note 1. If l, m, n be a principal direction corresponding to any root λ of the discriminating cubic, then we may easily see that the equation of the corresponding principal plane, takes the form

$$\lambda(lx+my+nz)+(ul+vm+wn)=0.$$

This equation shows that we can have no principal plane corresponding to $\lambda=0$ if $\lambda=0$ is a root of the discriminating cubic. In spite of this, however, we shall find it useful to say that l, m, n is a principal direction corresponding to $\lambda=0$. This every direction l, m, n satisfying the equations (1), (2), (3) corresponding to a root λ of the discriminating cubic (4) will be called a principal direction.

Note 2. In the following, we shall prove three important results concerning the nature of the root of the discriminating cubic and the corresponding principal directions.

11·41. Theorem I. *The roots of the discriminating cubic are all real.*

Suppose that λ is any root of the discriminating cubic (4) and l, m, n any non-zero set of values of l, m, n satisfying the corresponding equations (1), (2), (3).

Here it should be remembered that we cannot regard l, m, n as real, for λ is not yet proved to be real.

In the following, the complex conjugate of any number will be expressed by putting a bar over the same. Thus $\bar{l}$, $\bar{m}$, $\bar{n}$ will denote the complex conjugates of l, m, n respectively.

Now we have

$$al+hm+gn=l\lambda,$$
$$hl+bm+fn=m\lambda,$$
$$gl+fm+cn=n\lambda.$$

Multiplying these by $\bar{l}, \bar{m}, \bar{n}$ respectively and adding, we obtain

$$\Sigma a l \bar{l} + \Sigma f(m\bar{n} + \bar{m} n) = \lambda \Sigma l \bar{l} \quad \ldots(5)$$

Now a, b, c, f, g, h are all real. Also $l\bar{l}, m\bar{m}, n\bar{n}$ being the products of pairs of conjugate complex numbers, are real. Further we notice that $\bar{m} n$ is the conjugate complex of $m\bar{n}$ so that

$$m\bar{n} + \bar{m} n$$

is real.

Similarly

$$n\bar{l} + \bar{n} l, \quad l\bar{m} + \bar{l} m$$

are real.

Finally $\Sigma l \bar{l}$ is a non-zero real number.

Thus λ, being the ratio of two real numbers from (5), is necessarily a real number.

Hence the roots of the discriminating cubic are all real. Also, therefore, l, m, n corresponding to each λ are real.

11·42. Theorem II. *The two principal directions corresponding to any two distinct roots of the discriminating cubic are perpendicular.*

Suppose that

$$\lambda_1, \lambda_2$$

are two distinct roots, and

$$l_1, m_1, n_1 \; ; \; l_2, m_2, n_2$$

are the two corresponding principal directions.

We then have

(6) $al_1 + hm_1 + gn_1 = \lambda_1 l_1$, (9) $al_2 + hm_2 + gn_2 = \lambda_2 l_2$,

(7) $hl_1 + bm_1 + fn_1 = \lambda_1 m_1$, (10) $hl_2 + bm_2 + fn_2 = \lambda_2 m_2$,

(8) $gl_1 + fm_1 + cn_1 = \lambda_1 n_1$, (11) $gl_2 + fm_2 + cn_2 = \lambda_2 n_2$.

Multiplying (6), (7), (8) by l_2, m_2, n_2 respectively and adding we obtain

$$\Sigma a l_1 l_2 + \Sigma f(m_1 n_2 + m_2 n_1) = \lambda_1 \Sigma l_1 l_2 \quad \ldots(12)$$

Also multiplying (9), (10), (11) by l_1, m_1, n_1 respectively and adding, we obtain

$$\Sigma a l_2 l_1 + \Sigma f(m_1 n_2 + m_2 n_1) = \lambda_2 \Sigma l_1 l_2 \quad \ldots(13)$$

From (12) and (13), we obtain

$$\lambda_1 \Sigma l_1 l_2 = \lambda_2 \Sigma l_1 l_2$$

so that

$$(\lambda_1 - \lambda_2) \Sigma l_1 l_2 = 0.$$

Hence

$$\Sigma l_1 l_2 = 0, \qquad \text{for } \lambda_1 - \lambda_2 \neq 0.$$

Thus the two directions are perpendicular. Hence the theorem.

11·43. Theorem III. *For every quadric, there exists at least one set of three mutually perpendicular principal directions.*

We have to consider the following three cases :

(A) When the roots of the discriminating cubic are all distinct.

(B) When two roots are equal and the third is different from these.

(C) When the three roots are all equal.

These three cases will be considered one by one.

(A) The roots being distinct, there will correspond a principal direction l, m, n satisfying (1), (2), (3) on page 230 to each of these and by theorem II, these three directions will be mutually perpendicular. The three principal directions are unique in this case.

(B) Let λ be a root of the discriminating cubic repeated twice so that besides satisfying (4), on page 230 *viz.*,

$$\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0 \qquad \ldots(4)$$

it also satisfies

$$3\lambda^2-2\lambda(a+b+c)+(A+B+C)=0, \qquad \ldots(14)$$

which is obtained on differentiating the cubic *w.r.t.* λ.

We can re-write (14) as

$$[(b-\lambda)(c-\lambda)-f^2]+[(c-\lambda)(a-\lambda)-g^2]+[(a-\lambda)(b-\lambda)-h^2]=0.$$

It has been shown in § 11·23, p. 227 that if

$$D=0,\ A+B+C=0,$$

then

$$A, B, C, F, G, H,$$

are all zero.

Since, as may be easily seen, the relations (4) and (14) above can be obtained on replacing

$$a, b, c \text{ by } a-\lambda, b-\lambda, c-\lambda$$

respectively in the relations

$$D=0,\ A+B+C=0,$$

we see that corresponding to the vanishing of A, B, C, F, G, H, we have here

$$\left.\begin{aligned}(b-\lambda)(c-\lambda)=f^2,\quad (c-\lambda)(a-\lambda)=g^2,\quad (a-\lambda)(b-\lambda)=h^2\ ;\\ (a-\lambda)f=gh,\qquad (b-\lambda)g=hf,\qquad (c-\lambda)h=fg.\end{aligned}\right] \qquad \ldots(15)$$

These relations show that the equations (1), (2), (3) on page 230 for the determination of l, m, n, corresponding to λ are all equivalent. [Refer § 11·24, page 228]

Thus we see that *if* λ *is a twice repeated root, then every direction* l, m, n *satisfying the single relation,*

$$(a-\lambda)l+hm+gn=0 \qquad \ldots(16)$$

[or any equivalent relation (2), (3)]

is a principal direction corresponding to λ.

Suppose now that l, m, n is* any direction satisfying (16). Further we determine a direction l_2, m_2, n_2 satisfying (16) and perpendicular to l_1, m_1, n_1. Thus l_2, m_2, n_2 are determined from

$$(a-\lambda)l_2+hm_2+gn_2=0,$$
$$l_1l_2+m_1m_2+n_1n_2=0.$$

*If desired, l_1, m_1, n_1 may be selected further so as to satisfy some additional suitable condition.

Thus we have obtained two perpendicular principal directions l_1, m_1, n_1 ; l_2, m_2, n_2 corresponding to the twice repeated root λ.

Also let l_3, m_3, n_3 be the principal direction corresponding to the third root λ_3. By Theorem II, this direction l_3, m_3, n_3, will be perpendicular to each of the two perpendicular principal directions l_1, m_1, n_1 ; l_2, m_2, n_2.

Thus the theorem is true in this case.

Note. It is important to notice that in the case every direction perpendicular to the principal direction corresponding to the non-repeated root λ_3 is a principal direction for the twice-repeated root λ.

(C) Suppose now that all the three roots are equal to λ.

In this case λ satisfies the three equations

$$\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0, \quad \ldots(4)$$

$$3\lambda^2-2\lambda(a+b+c)+(A+B+C)=0, \quad \ldots(14)$$

$$3\lambda-(a+b+c)=0 \quad \ldots(17)$$

In this case also the relation (15), page 232 as deduced from (4) and (14) are true.

We re-write (17) as

$$(a-\lambda)+(b-\lambda)+(c-\lambda)=0.$$

Also by (15),

$$(b-\lambda)(c-\lambda)=f^2,\ (c-\lambda)(a-\lambda)=g^2,\ (a-\lambda)(b-\lambda)=h^2$$

so that

$$a-\lambda,\ b-\lambda,\ c-\lambda$$

must all have the same sign. Thus as in § 11·23, page 228, we deduce that

$$a-\lambda=0,\ b-\lambda=0,\ c-\lambda=0$$

so that

$$\lambda=a=b=c.$$

Also then it follows that $f=0$, $g=0$, $h=0$.

We now see that in this case the equations

$$(a-\lambda)l+hm+gn=0,\ hl+(b-\lambda)m+fn=0,\ gl+fm+(c-\lambda)n=0$$

for the determination of the principal direction are identically satisfied, *i.e.*, they are true for every value of l, m, n, so that every direction is a principal direction.

Thus in this case also a quadric has a set of three mutually perpendicular principal directions. In fact, any set of three mutually perpendicular directions is a set of three mutually perpendicular principal directions in this case.

The reader may observe that the quadric is a sphere in this last case.

Examples

Find a set of three mutually perpendicular principal directions for the following conicoids :

1. $3x^2+5y^2+3z^2-2yz+2zx-2xy+2z=0.$

2. $8x^2+7y^2+3z^2-8yz+4zx-12xy+2x-8y+1=0.$

3. $6x^2+3y^2+3z^2-2yz+4zx-4xy-3y+5z=0.$

1. We have

$$a=3,\ b=5,\ c=3,\ f=-1,\ g=1,\ h=\ \ 1.$$

Therefore the discriminating cubic is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}=0,$$

i.e.,

$$-\lambda^3+11\lambda^2-36\lambda+36=0..$$

Its roots are

$$\lambda=2,\ 3,\ 6.$$

The principal direction corresponding to $\lambda=2$ is given by

$$\begin{aligned} l-m+n&=0,\\ -l+3m-n&=0,\\ l-m+n&=0. \end{aligned}$$

These give

$$l:m:n=1:0:-1.$$

Thus the principal direction corresponding to $\lambda=2$ is given by

$$\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}.$$

Again the principal direction corresponding to $\lambda=3$ is given by

$$\begin{aligned} 0l-m+n&=0,\\ -l+2m-n&=0,\\ l-m+0n&=0, \end{aligned}$$

so that

$$l:m:n=1:1:1,$$

and we have the principal direction

$$\frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}},\ \frac{1}{\sqrt{3}}.$$

Finally the principal direction corresponding to $\lambda=6$ is given by

$$\begin{aligned} -3l-m+n&=0,\\ -l-m-n&=0,\\ l-m-3n&=0, \end{aligned}$$

wherefrom we may see that this principal direction is

$$\left(\frac{1}{\sqrt{6}},\ -\frac{2}{\sqrt{6}},\ \frac{1}{\sqrt{6}}\right).$$

The principal plane corresponding to λ being

$$\lambda(lx+my+nz)+(ul+vm+wn)=0,$$

we may see that the three principal planes are

$$\begin{aligned} 2x-2z-1&=0,\\ 3x+3y+3z+1&=0,\\ 6x-12y+6z+1&=0. \end{aligned}$$

2. We have

$$a=8,\ b=7,\ c=3,\ f=-4,\ g=2,\ h=-6.$$

Therefore the discriminating cubic is

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0,$$

i.e.,

$$-\lambda^3+18\lambda^2-45\lambda=0.$$

$$\therefore \quad \lambda=0,\ 3,\ 15.$$

Thus 0, 3, 15 are three distinct characteristic roots.

The principal direction l, m, n corresponding to $\lambda=0$ is given by

$$8l-6m+2n=0,$$
$$-6l+7m-4n=0,$$
$$2l-4m+3n=0.$$

Solving these, we see that

$$l : m : n=1 : 2 : 2.$$

Thus the principal direction corresponding to $\lambda=0$ is given by

$$\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{2}{3}.$$

Again the principal direction corresponding to $\lambda=3$ is given by

$$5l-6m+2n=0,$$
$$-6l+4m-4n=0,$$
$$2l-4m+0n=0.$$

These give

$$l : m : n=2 : 1 : -2,$$

so that the corresponding principal direction is given by

$$\tfrac{2}{3},\ \tfrac{1}{3},\ -\tfrac{2}{3}.$$

Finally the principal direction corresponding to $\lambda=15$ is given by

$$-7l-6m+2n=0,$$
$$-6l-8m-4n=0,$$
$$2l-4m-12n=0.$$

which give

$$l : m : n=2 : -2 : 1,$$

so that the corresponding principal direction is given by

$$\tfrac{2}{3},\ -\tfrac{2}{3},\ \tfrac{1}{3}.$$

The reader may verify that the three directions are mutually perpendicular.

The principal plane corresponding to λ being

$$\lambda(lx+my+nz)+(ul+vm+wn)=0,$$

we may see that the two principal planes corresponding to the non-zero values 3, 15 of λ are

$$3(2x+y-2z)+(-2)=0, \text{ i.e., } 6x+3y-6z-2=0$$

and

$$15(2x-2y+z)+10=0, \text{ i.e., } 6x-6y+3z+2=0$$

3. We have

$$a=6,\ b=3,\ c=3,\ f=-1,\ g=2,\ h=-2.$$

The discriminating cubic is

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0,$$

i.e.,

$$-\lambda^3+12\lambda^2-36\lambda+32=0,$$

whose roots are 2, 2, 8. Thus two roots are equal. Firstly we consider the non-repeated root 8. The principal direction corresponding to this is given by

$$\begin{aligned} -2l-2m+2n&=0,\\ -2l-5m-\ 2&=0,\\ 2l-\ m-5n&=0. \end{aligned}$$

These give

$$l : m : n=2 : -1 : 1$$

so that the principal direction corresponding to $\lambda=8$ is given by

$$\frac{2}{\sqrt{6}},\quad -\frac{1}{\sqrt{6}},\quad \frac{1}{\sqrt{6}}.$$

Again the principal direction corresponding to $\lambda=2$ is given by

$$\begin{aligned} 4l-2m+2n&=0,\\ -2l+\ m-\ n&=0,\\ 2l-\ m+\ n&=0. \end{aligned}$$

It is easily seen that these three equations for the determination of l, m, n are all equivalent. This fact had been generally established in Theorem III for principal directions corresponding to twice repeated characteristic roots.

Thus every l, m, n satisfying the single equation

$$2l-m+n=0 \qquad \ldots(1)$$

determines a principal direction. Consider any set of values of l, m, n, satisfying (1) say

$$-1,\ -1,\ 1.$$

We write

$$l_1 : m_1 : n_1=-1 : -1 : 1.$$

Then we determine l_2, m_2, n_2 satisfying (1) and perpendicular to l_1, m_1, n_1.

Thus

$$2l_2-m_2+n_2=0.$$
$$-l_2-m_2+n_2=0.$$

These give

$$l_2 : m_2 : n_2=0 : 1 : 1.$$

Thus we have obtained a set of three mutually perpendicular principal directions given by

$$\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}; \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}.$$

The choice of principal directions is not unique in the present case as two characteristic roots are equal.

Note. It may be verified that every direction perpendicular to the principal direction corresponding to the non-repeated root 8 is a principal direction for the twice repeated root 2. [Refer note at the end of § 11·43 *B*, page 231.]

Exercises

Examine the following quadrics for principal directions and principal planes.

1. $4x^2-y^2-z^2+2yz-8x-4y+8z=0.$
2. $x^2+2yz-4x+6y+2z=0.$
3. $4y^2-4yz+4zx-4xy-2x+2y-1=0.$
4. $3x^2-y^2-z^2+6yz-6x+6y-2z-2=0.$

Answers

1. Principal directions : 1, 0, 0 ; 0, $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$; 0, $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.
   Principal planes : $x=1$, $y-z+3=0$.
2. Principal directions ; $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and every direction perpendicular to it.
   Principal planes : $y-z-2=0$ and any plane through the line, $y+z+4=0$, $x=2$.
3. Principal directions $\frac{1}{\sqrt{3}}, \frac{1}{3}, \frac{1}{\sqrt{3}}$; $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$; $\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$.
   Principal planes. Any plane at right angle to $x=y=z-\frac{1}{3}$;
   $2(x-2y+z)=1$, $2(x-z)+1=0$.
4. Principal directions : $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$; 1, 0, 0 ; 0, $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$.
   Principal planes ; $y+z+1=0$, $x=1$, $y-z=1$.

11·5. Centre. We know that if a point (x, y, z) is the midpoint of a chord with direction cosines l, m, n of a quadric

$$F(x, y, z)=0,$$

then we have

$$l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0. \qquad \ldots(1)$$

This shows that if (x, y, z) is such that

$$\frac{\partial F}{\partial x}=0, \quad \frac{\partial F}{\partial y}=0, \quad \frac{\partial F}{\partial z}=0,$$

then the condition (1) is satisfied, whatever values l, m, n may have, *i.e.*, every chord through (x, y, z) is bisected thereat. Such a point is known as a *Centre* of the quadric. We can re-write these as

$$ax+hy+gz+u=0, \quad \ldots(2)$$
$$hx+by+fz+v=0, \quad \ldots(3)$$
$$gx+fy+cz+w=0. \quad \ldots(4)$$

It should be remembered that a quadric may or may not have a centre ; also it may have more than one centre — *a line of centres* or *a plane of centres*, depending upon the nature of the solutions of the three equations (2), (3), (4).

In the following, we shall consider the different cases regarding the possible solutions of these equations. This discussion will be facilitated a good deal, if, regarding x, y, z as variables, we consider the three planes represented by these equations. We have thus to examine the nature of the points of intersection, if any, of these three planes to be called *Central planes.*

11·51. Case of Unique Centre. Multiplying the equations (2), (3), (4) by A, H, G, respectively and adding, we obtain

$$Dx+(Au+Hv+Gw)=0 \quad \text{(Refer § 11·21, page 227]}$$

Again, on multiplying (2), (3), (4) by H, B, F, and by G, F, C and adding separately we obtain

$$Dy+(Hu+Bv+Fw)=0,$$
$$Dz+(Gu+Fv+Cw)=0.$$

If $D\neq 0$, we obtain from these

$$x=-(Au+Hv+Gw)/D,$$
$$y=-(Hu+Bv+Fw)/D,$$
$$z=-(Gu+Fv+Cw)/D.$$

Substituting these in (2), (3), (4) we may easily verify that the same are satisfied.

Thus if $D\neq 0$, the quadric has a unique centre (x, y, z) where (x, y, z) have the values given above.

11·52. *Now suppose that $D=0$.* Then, we have,

$$A(ax+hy+gz+u)+H(hx+by+fz+v)+G(gx+fy+cz+w) \equiv Au+Hv+Gw.$$

[Refer § 11·21, p. 226].

This shows that the three equations cannot have a common solution, *i.e.*, the quadric will not have a centre if

$$Au+Hv+Gw\neq 0.$$

Considering H, B, F and G, F, C as sets of multipliers instead of A, H, G, we may similarly see that the quadric will not have a centre if

$$Hu+Bv+Fw\neq 0 \text{ or if } Gu+Fv+Cw\neq 0.$$

Thus we see that *the quadric will not have a centre if $D=0$ and any one of*

$$Au+Hv+Gw,\ Hu+Bv+Fw,\ Gu+Fv+Cw$$

is not zero.

11·53. We now suppose that

$$D=0 \text{ as well as } Au+Hv+Gw=0.$$

Then we have

$$A(ax+hy+gz+u)+H(hx+by+fz+v)+G(gx+fy+cz+w)\equiv 0.$$

(*i*) Thus if $A\neq 0$, we have

$$ax+hy+gz+u=-\frac{H}{A}(hx+by+fz+v)-\frac{G}{A}(gx+fy+cz+w).$$

(*ii*) Also if $A\neq 0$, the two planes

$$hx+by+fz+v=0$$
$$gx+fy+cz+w=0,$$

are neither same nor parallel so that they intersect in a line. This is because

$$\begin{vmatrix} b & f \\ f & c \end{vmatrix} = A\neq 0. \quad \text{[Refer § 11·24, p. 228]}$$

From (*i*) and (*ii*), we deduce that the plane

$$ax+hy+gz+u=0$$

passes through the line of intersection of the two intersecting planes

$$hx+by+fz+v=0,\ gx+fy+cz+w=0.$$

Thus in case

$$D=0,\ Au+Hv+Gw=0,\ A\neq 0,$$

the three central planes all pass through one line so that we have a line of centres.

We may similarly see that the quadric will have a line of centres

if $$D=0,\ Hu+Bv+Fw=0,\ B\neq 0,$$

or if $$D=0,\ Gu+Fv+Cw=0,\ C\neq 0,$$

Note 1. We can show that if $D=0$ and $A\neq 0$ and $Au+Hv+Gw=0$, then we must also simultaneously have

$$Hu+Bv+Fw=0,\ Gu+Fv+Gw=0.$$

In fact we have

$$A(Hu+Bv+Fw)\equiv H(Au+Hv+Gw)$$

and

$$A(Gu+Fv+Cw)\equiv G(Au+Hv+Cw),$$

the equalities holding for all values of u, v and w. Thus if $A\neq 0$, we have

$$Hu+Bv+Fw=\frac{H}{A}(Au+Hv+Gw),$$

$$Gu+Fv+Cw=\frac{G}{A}(Au+Hv+Gw).$$

The result stated now follows.

It may be remembered that if $A=0$ then also $H=0$, $G=0$, so that $Au+Hv+Gw\equiv 0$. In this case when $A=0$, $H=0$, $G=0$, we may not have

$$Hu+Bv+Fw=0 \text{ or } Gu+Fv+Cw=0.$$

For example consider

$$x^2+2y^2+2xy+2x-y+2z+3=0,$$

Here

$$a=1,\ b=2,\ c=0,\ f=0,\ g=0,\ h=1,\ u=1,\ v=\tfrac{1}{2},\ w=1,$$

so that

$$A=0,\ B=0.\ C=1,\ F=0,\ G=0,\ H=0,\ D=0.$$

Thus we have $Au+Hv+Gw=0$ but $Gu+Fv+Cw\neq 0$.

Note 2. The cases treated above in §§ 11·52 and 11·53 cover the cases when $D=0$ and one at least of A, B, C is not zero.

If we suppose that A, B, C are all zero, then it follows that F, G, H are also all zero, for

$$F^2=BC,\ G^2=CA,\ H^2=AB.$$

In the next sub-section we consider the case when A, B, C, F, G, H are all zero. The vanishing of D then follows from the vanishing of these co-factors in as much as we have

$$D=Aa+Hh+Gg,$$

so that $D=0$ even if A, H, G only are known to be zero.

11·54. *Suppose now that A, B, C, F, G, H are all zero.*

In this case we have $D=0$ also.

We have, in this case,

$$(1)\quad \begin{cases} f(ax+hy+gz+u)-g(hx+by+fz+v)\equiv fu-gv, \\ f(ax+hy+gz+u)-h(gx+fy+cz+w)\equiv fu-hw. \end{cases}$$

These show that if

$$fu-gv\neq 0 \text{ or } fu-hw\neq 0,$$

then the quadric cannot have a centre.

11·55. *Suppose now that*

$$fu-gv=0 \text{ and } fu-hw=0,$$

i.e.,

$$fu=gv=hw.$$

Then if $g\neq 0$, $h\neq 0$, we have, from (1) above in § 11·54 that

$$hx+by+fz+v\equiv\frac{f}{g}(ax+hy+gz+u),$$

$$gx+fy+cz+w\equiv\frac{f}{h}(ax+hy+gz+u),$$

so that every point of the plane

$$ax+hy+gz+u=0$$

is also a point of the other two central planes. Thus we have a plane of centres in this case.

Similarly we may show that if

$$fu=gv=hw$$

and some two of f, g, h are not zero, then the quadric has a plane of centres.

Note. It can be easily seen that if A, B, C, F, G, H are zero and one of f, g, h is known to be zero, then one more of f, g, h must also be zero. For instance, suppose that $f=0$. Then, because,

$$0=F=gh-af,$$

it follows that either g or h must also be zero. Thus the case treated here can be stated as follows:

If A, B, C, F, G, H are all zero, none of f, g, h is zero and $fu=gv=hw$, then the quadric has a line of centres.

The case where one and, therefore, two of f, g, h are zero is treated here below.

11·56. Now *suppose that two of* f, g, h *are zero* in addition to A, B, C, F, G, H being all zero and $fu=gv=hw$. Let $g=0=h$ and $f\neq 0$. In this case we see from (1) above, § 11·54, p. 240 that

$$ax+hy+gz+u\equiv 0$$

so that $a=0$, $h=0$, $g=0$, $u=0$.

The vanishing of u also follows from the fact that

$$fu=gv=hw \text{ and } g=0,\ h=0, f\neq 0.$$

Consider now the two central planes

$$hx+by+fz+v=0,$$
$$gx+fy+cz+w=0,$$

the co-efficients of the third central plane being all zero. As h and g are both zero, we can re-write these as

$$by+fz+v=0,$$
$$fy+cz+w=0.$$

Here

$$\begin{vmatrix} b & f \\ f & c \end{vmatrix} = bc-f^2=A=0, \quad \begin{vmatrix} f & v \\ c & w \end{vmatrix} = fw-cv.$$

Thus, if $fw-cv\neq 0$, *the quadric has no centre and if* $fw-cv=0$, *the quadric has a plane of centres.*

We can obtain similar conditions when

$$f=0=h,\ g\neq 0$$

or when

$$f=0=g,\ h\neq 0.$$

11·57. Now suppose that f, g, h are all zero in addition to the vanishing of A, B, C, F, G, H.

In this case two of a, b, c must be zero. Suppose that $b=c=0$ and $a\neq 0$. Then the first of the three central planes is

$$ax+u=0$$

and the other two are

$$0x+0y+0z+v=0$$
$$0x+0y+0z+w=0.$$

Thus if $v\neq 0$ or $w\neq 0$ the quadric has no centre and if $v=0=w$, the quadric has a plane of centres.

Summary of the various cases

1. $D\neq 0$. Unique centre.

2.
$$\begin{cases} D=0,\ Au+Hv+Gw\neq 0. & \text{No centre.} \\ D=0,\ Hu+Bv+Fw\neq 0. & \text{No centre.} \\ D=0,\ Gu+Fv+Cw\neq 0. & \text{No centre.} \end{cases}$$

3.
$$\begin{cases} D=0,\ Au+Hv+Fw=0.\ A\neq 0, & \text{Line of centres.} \\ D=0,\ Hu+Bv+Gw=0.\ B\neq 0, & \text{Line of centres.} \\ D=0,\ Gu+Fv+Cw=0.\ C\neq 0, & \text{Line of centres.} \end{cases}$$

4. A, B, C, F, G, H all zero and $fu \neq gv$ or $gv \neq hw$. No centre.

5. A, B, C, F, G, H all zero, $fu=gv=hw$, $f \neq 0$, $g \neq 0$, $h \neq 0$. Plane of centres.

6. A, B, C, F, G, H all zero, $fu=gv=hw$, $g=0, h=0, f \neq 0, fw-cv \neq 0$. No centre.

7. A, B, C, F, G, H all zero, $fu=gv=hw$, $g=0$, $h=0, f \neq 0, fw-cv=0$ Plane of centres.

We may have results similar to (6) and (7) when $f=0$, $g=0$, $h \neq 0$ or when $h=0, f=0$, $g \neq 0$.

8. A, B, C, F, G, H all zero; f, g, h all zero. Then two of a, b, c must be zero and one none-zero. Then we have no centre if

$$a \neq 0, \ v \neq 0, \text{ or } w \neq 0$$

and a plane of centres if

$$a \neq 0, \ v=0=w.$$

We have similar results when $b \neq 0$ or $c \neq 0$.

Note. The results given above need not be committed to memory.

Exercises

Examine the following quadrics for centre :

1. $z^2-yz+zx+xy-2y+2z+2=0$.

[*Ans.* Unique centre (1, 1, −1).

2. $2z^2-2yz-2zx+2xy+3x-y-2z+1=0$.

[*Ans.* Line of centres ; $\frac{x}{1}=\frac{y+2}{1}=\frac{2z+1}{2}$.

3. $4x^2+9y^2+4z^2+12xy+12yz+8zx+3x+4y+z=0$.

[*Ans.* No centre.

4. $x^2+y^2+z^2-2xy-2yz+2zx+x-y+z=0$.

[*Ans.* Plane of centres ; $2x-2y+2z+1=0$.

5. $4x^2-2y^2-2z^2+5yz+2zx+2xy-x+2y+2z-1=0$.

[*Ans.* No centre.

6. $2x^2+2y^2+5z^2-2yz-2zx-4xy-14x-14y+16z+6=0$.

[*Ans.* Line of centres ; $x=3-y$, $z+1=0$.

7. $18x^2+2y^2+20z^2-12zx+12yz+2x-22y-6z+1=0$.

[*Ans.* No centre,

8. $4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$.

[*Ans.* Unique centre : (−1, −2, −3)

11·6. Transformation of Co-ordinates. Before we take up the problem of the actual reduction and classification, we shall consider two important cases of transformation of co-ordinates.

11·61. The form of the equation of a quadric referred to centre as origin. We suppose that the given quadric has a centre. Let (α, β, γ) be the centre of the quadric with equation,

$$F(x, y, z) \equiv \Sigma(ax^2+2fyz)+2\Sigma ux+d=0.$$

Consider now a new system of co-ordinate axes parallel to the given system and with its origin at (α, β, γ). The equation of the quadric *w. r. t.* the new system, obtained on replacing x, y, z by $x+\alpha$, $y+\beta$, $z+\gamma$ respectively, is

$$\Sigma[a(x+\alpha)^2+2f(y+\beta)(z+\gamma)]+2\Sigma u(x+\alpha)+d=0,$$

i.e.,

$$\Sigma(ax^2+2fyz)+2x(a\alpha+h\beta+g\gamma+u)+2y(h\alpha+b\beta+f\gamma+v)$$
$$+2z(g\alpha+f\beta+c\gamma+w)+F(\alpha, \beta, \gamma)=0.$$

As (α, β, γ) is a centre, we have

$$a\alpha+h\beta+g\gamma+u=0,\ h\alpha+b\beta+f\gamma+v=0,\ g\alpha+f\beta+c\gamma+w=0.$$

Further, as may be easily seen,

$$F(\alpha, \beta, \gamma)=\alpha(a\alpha+h\beta+g\gamma+u)+\beta(h\alpha+b\beta+f\gamma+v)$$
$$+\gamma(g\alpha+f\beta+c\gamma+w)+(u\alpha+v\beta+w\gamma+d)$$
$$=u\alpha+v\beta+w\gamma+d.$$

Thus the required new equation is

$$\Sigma(ax^2+2fyz)+(u\alpha+v\beta+w\gamma+d)=0.$$

so that the second degree homogeneous part has remained unchanged.

Note 1. The discussion above is applicable whether the quadric has one centre, a line of centres or a plane of centres. In case the quadric has more than one centre, then (α, β, γ) may denote any one of the centres.

Note 2. The co-ordinates *w. r. t.* the old as well as new systems of axes has both been denoted by the same symbols. x, y, z.

11·62. The form of the equation of a quadric, when the co-ordinate axes are parallel to a set of three mutually perpendicular principal directions. Suppose that

$$l_1, m_1, n_1, ; l_2, m_2, n_2, l_3, m_3, n_3 \qquad \ldots(1)$$

are the direction cosines of three mutually perpendicular principal directions corresponding to the three roots

$$\lambda_1, \lambda_2, \lambda_3$$

of the discriminating cubic. Here one or two of these roots may be zero.

We take now a new co-ordinate system through the same origin such that the axes of the new system are parallel to the directions given by (1) above.

The equation referred to the new system of axes is obtained on replacing

$$x, y, z$$

by

$$l_1x+l_2y+l_3z,\ m_1x+m_2y+m_3z,\ n_1x+n_2y+n_3z$$

respectively.

As homogeneous linear expressions are to be substituted for x, y, z, we may note that a homogeneous expression of any degree will be transformed to a homogeneous expression of the same degree.

Thus we may separately consider the transforms of the homogeneous parts

$$\Sigma(ax^2+2fyz) \text{ and } 2\Sigma ux.$$

We shall now prove a very important result, viz., that the transform of

$$\Sigma(ax^2+2fyz) \qquad \ldots(1)$$

is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2.$$

On direct substitution, we may see that the co-efficient of x^2 in the transform of (1) is

$$al_1^2+bm_1^2+cn_1^2+2fm_1n_1+2gn_1l_1+2hl_1m_1$$
$$=l_1(al_1+hm_1+gn_1)+m_1(hl_1+bm_1+fn_1)+n_1(gl_1+fm_1+cn_1)$$
$$=l_1(\lambda_1 l_1)+m_1(\lambda_1 m_1)+n_1(\lambda_1 n_1) \qquad \text{[by § 11·4, p. 229]}$$
$$=\lambda_1(l_1^2+m_1^2+n_1^2)=\lambda_1.$$

Similarly the co-efficients of y^2 and z^2 in the transform can be shown to be

$$\lambda_2 \text{ and } \lambda_3$$

respectively.

Again the co-efficient of $2yz$ in the transform of (1)

$$=al_2l_3+bm_2m_3+cn_2n_3+f(m_2n_3+m_3n_2)+g(n_2l_3+n_3l_2)+h(l_2m_3+l_3m_2)$$
$$=l_2(al_3+hm_3+gn_3)+m_2(hl_3+bm_3+fn_3)+n_2(gl_3+fm_3+cn_3)$$
$$=\lambda_3(l_2l_3+m_2m_3+n_2n_3)=0.$$

Similarly the co-efficients of zx and xy in the transform can be seen to be zero.

Thus the transform of

$$\Sigma(ax^2+2fyz)$$

is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2.$$

Finally we see that the transform of

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d$$

is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+2u(l_1x+l_2y+l_3z)+$$
$$2v(m_1x+m_2y+m_3z)+2w(n_1x+n_2y+n_3z)+d$$
$$=\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+2x(ul_1+vm_1+wn_1)$$
$$+2y(ul_2+vm_2+wn_2)+2z(ul_3+vm_3+wn_3)+d.$$

11·7. Reduction to canonical forms and classification.

We shall now consider the several cases one by one.

11·71. Case I. *When $D\neq 0$. In this case the quadric has a unique centre and no root of the discriminating cubic is zero.*

Shifting the origin to the centre (α, β, γ), the equation takes the form

$$\Sigma(ax^2+2fyz)+(u\alpha+v\beta+w\gamma+d)=0. \qquad (\S\ 11\cdot61 \text{ p. } 242)$$

Now rotating the axes so that the axes of the new system are parallel to the set of three mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+(u\alpha+v\beta+w\gamma+d)=0,$$

which is the required canonical form.

Below we shall find an elegant form for the constant term.

We have

$$a\alpha+h\beta+g\gamma+u=0, \qquad \ldots(1)$$
$$h\alpha+b\beta+f\gamma+v=0, \qquad \ldots(2)$$
$$g\alpha+b\beta+c\gamma+d=0. \qquad \ldots(3)$$

Also we write

$$ux+vy+wz+d=k,$$

i.e.,
$$ux+vy+wz+(d-k)=0. \qquad \ldots(4)$$

Eliminating x, y, z from (1), (2), (3) and (4) we obtain

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & d \\ u & v & w & (d-k) \end{vmatrix} = 0$$

i.e.,

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} - k \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\therefore \qquad k=\frac{\triangle}{D}, \qquad (D\neq 0)$$

where we have represented the fourth order determinant on the left by $\triangle$.

Finally, therefore, the equation assumes the form

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+\frac{\triangle}{D}=0.$$

This equation represents various types of surfaces as shown in the following table. It may be remembered that the word 'roots' refers to characteristic roots.

$\triangle=0$	roots all >0 or <0	Imaginary cone.
$\triangle=0$	Two roots >0 and one <0	Real cone.
$\triangle=0$	Two roots <0 and one >0	Real cone.
$\triangle/D>0$	Roots all >0	Imaginary ellipsoid.
$\triangle/D>0$	Roots all <0	Real ellipsoid.
$\triangle/D>0$	Two roots >0 and one <0	Hyperboloid of two sheets.
$\triangle/D>0$	Two roots <0 and one >0	Hyperboloid of one sheet.
$\triangle/D<0$	Roots all >0	Real ellipsoid.
$\triangle/D<0$	Roots all <0	Imaginary ellipsoid.
$\triangle/D<0$	Two roots >0 and one <0	Hyperboloid of one sheet.
$\triangle/D<0$	Two roots <0 and one >0	Hyperboloid of two sheets.

11·72. Case II. *When* $D=0$, $Au+Hv+Gw\neq 0$. *In this case the quadric has no centre and the discriminating cubic has one zero root and two non-zero-roots.*

We denote the non-zero roots by λ_1, λ_2. The third root $\lambda_3=0$.

We rotate the co-ordinate axes through the same origin so that new axes are parallel to the set of three mutually perpendicular principal directions.

The new equation takes the form

$$\lambda_1 x^2+\lambda_2 y^2+2x(ul_1+vm_1+wn_1)+2y(ul_2+vm_2+wn_2)+2z(ul_3+vm_3+wn_3)+d=0; \qquad \ldots(1)$$

l_3, m_3, n_3 corresponding to $\lambda_3=0$.

Here we notice that

$$ul_3+vm_3+wn_3\neq 0.$$

Let, if possible,

$$ul_3+vm_3+wn_3=0. \qquad \ldots(2)$$

We also have

$$hl_3+bm_3+fn_3=0. \quad \ldots(3)$$

$$gl_3+fm_3+cn_3=0. \quad \ldots(4)$$

As l_3, m_3, n_3 are not all zero, we have from (2), (3), (4).

$$\begin{vmatrix} u & v & w \\ h & b & f \\ g & f & c \end{vmatrix}=0,$$

i.e., $$Au+Hv+Gw=0,$$

which is contradictory to the given condition.

Denoting the co-efficients of x, y, z by p, q, r, we re-write (1) as

$$\lambda_1x^2+\lambda_2y^2+2px+2qy+2rz+d=0 \text{ where } r\neq 0,$$

i.e., $$\lambda_1\left(x+\frac{p}{\lambda_1}\right)^2+\lambda_2\left(y+\frac{q}{\lambda_2}\right)^2+2r\left[z+\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}-\frac{q^2}{\lambda_2}\right)\right]=0$$

so that shifting the origin to the point

$$\left[-\frac{p}{\lambda_1}, -\frac{q}{\lambda_2}, -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}-\frac{q^2}{\lambda_2}\right)\right],$$

we see that the equation takes the form

$$\lambda_1x^2+\lambda_2y^2+2rz=0,$$

where $$r=ul_3+vm_3+wn_3\neq 0.$$

This is the required canonical form in the present case.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1, λ_2 are of the same or opposite signs.

Axis and vertex of a paraboloid. It is known that z-axis is the axis and $(0, 0, 0)$ is the vertex of the paraboloid

$$\lambda_1x^2+\lambda_2y^2+2rz=0.$$

Also the principal directions of the paraboloid are those of the co-ordinate axes; the principal direction corresponding to the characteristic root zero being that of z-axis and the principal direction corresponding to the non-zero roots λ_1, λ_2 being those of x-axis and y-axis respectively. Further it can be easily seen that the principal planes corresponding to the non-zero characteristic roots are the planes $x=0$, $y=0$ whose intersection z-axis is the axis of the paraboloid. Thus we have the following important and useful result :

The line of intersection of the principal planes corresponding to the non-zero characteristic roots is the axis and the point where it meets the paraboloid is the vertex. Also the axis is parallel to the principal direction corresponding to the characteristic root zero.

11·73. Case III. *When $D=0$, $Au+Hv+Gw=0$, $A\neq 0$. In this case the quadric has a line of centres and the discriminating cubic has one zero and two non-zero roots.*

We may see that $A+B+C\neq 0$, for if it were so, then we would have A, B, C all zero and the condition $A\neq 0$ would be contradicted.

Since $D=0$ and $A+B+C\neq 0$, the discriminating cubic would have only one zero root.

Let (α, β, γ) be any centre. Shifting the origin to (α, β, γ) and rotating the axes so that the new axes are parallel to the set of mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1 x^2+\lambda_2 y^2+(u\alpha+v\beta+w\gamma+d)=0,$$

which is the required canonical form.

We may, as follows, obtain an expression for the constant term in a form free from α, β, γ.

In this case the central planes all pass through one line.

*We select the two lines

$$hx+by+fz+v=0,$$
$$gx+fy+cz+w=0.$$

Now (α, β, γ) is any point satisfying these two equations. Taking $a=0$, we have

$$b\beta+f\gamma+v=0,$$
$$f\beta+c\gamma+w=0.$$

Also we write

$$v\beta+w\gamma+(d-k)=0.$$

These give

$$\begin{vmatrix} b & f & v \\ f & c & w \\ v & w & d-k \end{vmatrix}=0,$$

so that

$$k=\frac{1}{A}\begin{vmatrix} b & f & v \\ f & c & w \\ v & w & d \end{vmatrix}$$

Thus the required canonical form is

$$\lambda_1 x^2+\lambda_2 y^2+k=0.$$

The equation represents various types of surfaces as shown in the follow table :

$k=0$	Roots both >0 or <0	Imaginary pair of planes
$k=0$	One root >0 and other <0	Pair of intersecting planes
$k>0$	Roots both >0	Imaginary cylinder
$k>0$	Roots both <0	Elliptic cylinder
$k>0$	One root >0 and other <0	Hyperbolic cylinder
$k<0$	Roots both >0	Elliptic cylinder
$k<0$	Roots both <0	Imaginary cylinder
$k<0$	One root >0 and other <0	Hyperbolic cylinder

*These are so selected that they are not the same. The condition $A\neq 0$ ensures the non-sameness of these two planes.

Cor. 1. Axis of a cylinder. The z-axis is known to be the axis of the cylinder

$$\lambda_1 x^2+\lambda_2 y^2+k=0, \qquad k\neq 0.$$

As in the case of a paraboloid, we have the following result regarding the axis of a cylinder.

The axis of a cylinder is the line of intersection of the principal planes corresponding to the non-zero characteristic roots. Also it is parallel to the principal direction corresponding to the characteristic root zero. The axis is also the line of centres.

Cor. 2. Planes bisecting angles between two planes. It may be seen that planes bisecting angles between the two planes

$$\lambda_1 x^2+\lambda_2 y^2=0$$

are

$$x=0,\ y=0.$$

Thus we have that

The two principal planes corresponding to the two non-zero characteristic roots are the two bisecting planes.

Cor. 3. The equation

$$\Sigma(ax^2+2fyz)=0$$

will represent a pair of planes if $D=0$.

11·74. Case IV. *When A, B, C, F, G, H are all zero and $fu\neq gv$.* In this case the quadric has no centre and two roots of the discriminating cubic are zero and one non-zero.

We rotate the axes so that the new axes are parallel to the three mutually perpendicular principal directions. The new equation takes the form

$$\lambda_1 x^2+2x(ul_1+vm_1+wn_1)+2y(ul_2+vm_2+wn_2)+2z(ul_3+vm_3+wn_3)+d=0.$$

As the two roots λ_2, λ_3 are equal, both being 0, we know that l_2, m_2, n_2, is *any* direction satisfying

$$al+hm+gn=0. \qquad \ldots(1)$$

We suppose that l_2, m_2, n_2 are so chosen that these satisfy (1) and

$$ul_2+vm_2+wn_2=0. \qquad \ldots(2)$$

Then l_3, m_3, n_3, are chosen so as to satisfy (1) and

$$l_3l_2+m_3m_2+n_3n_2=0.$$

Denoting the co-efficients of x and z by p, r, we re-write the equation as

$$\lambda_1 x^2+2px+2rz+d=0, \qquad \ldots(3)$$

the co-efficient by y being zero by (2).

Again we re-write (3) as

$$\lambda_1\left[x+\frac{p}{\lambda_1}\right]^2+2rz+\left(d-\frac{p^2}{\lambda_1}\right)=0. \qquad \ldots(4)$$

Also we may see that $r \neq 0$ for otherwise the quadric will have a centre. Again, we re-write (4) as

$$\lambda_1\left(x+\frac{p}{\lambda_1}\right)^2+2r\left[z+\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}\right)\right]=0,$$

Shifting the origin to

$$\left[-\frac{p}{\lambda_1},\ 0,\ -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}\right)\right],$$

we see that the equation becomes

$$\lambda_1 x^2+2rz=0,$$

which is the required canonical form.

The equation represents a parabolic cylinder in this case.

11·75. Case V. *When A, B, C, F, G, H are all zero, $fu=gv=hw$, and no one of f, g, h is zero.*

In this case the quadric has a plane of centres and the discriminating cubic has two zero and one non-zero root.

Let (α, β, γ) be any centre. Shifting the origin to (α, β, γ) and rotating the axes so that the axes of the new system are parallel to a set of three mutually perpendicular principal directions, we see that the equation becomes

$$\lambda_1 x^2+(u\alpha+v\beta+w\lambda+d)=0.$$

The equation represents a pair of parallel or same planes.

Note. The case when some two or all of f, g, h are zero can be easily considered and it can be shown that we shall have a parabolic cylinder in case the quadric does not have a centre and a pair of parallel planes if the quadric has a plane of centres.

11·8. Quadrics of revolution. Firstly we shall prove a lemma concerning surfaces of revolution obtained on revolving a plane curve about an axis of co-ordinates.

Lemma. *The equation of a surface of revolution obtained on revolving a plane curve about x-axis is of the form*

$$\sqrt{(y^2+z^2)}=f(x).$$

Consider any surface of revolution obtained on revolving a curve about x-axis. Let the equations of the section of this surface by the plane $z=0$ be

$$y=f(x),\ z=0, \qquad \ldots(1)$$

If P be any point on the curve and M the foot of the perpendicular from P on x-axis, we have

$$OM=x,\ MP=y$$

so that we can re-write $y=f(x)$ as

$$MP=f(OM). \qquad \ldots(2)$$

Now this relation remains unchanged as the curve revolves about x-axis so that P describes a circle with M as its centre.

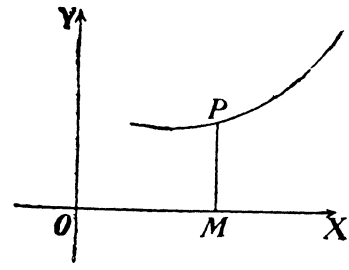

Fig. 31

In terms of the co-ordinates (x, y, z) of the point P in any position, we have

$$MP=\sqrt{y^2+z^2},\ OM=x$$

so that we can now re-write (2) as

$$\sqrt{(y^2+z^2)}=f(x).$$

Hence the result.

Similarly the equations of the surfaces of revolution obtained on revolving plane curves about y-axis and z-axis are of the form

$$\sqrt{(z^2+x^2)}=\phi\ (y),$$
$$\sqrt{(x^2+y^2)}=\psi\ (z)$$

respectively.

Cor. *A quadric is a surface of revolution, if and only if it has equal non-zero characteristic roots.* To see the truth of this result, we examine the various canonical forms which we have obtained. These are as follows :

Case I	$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+\triangle/D=0,$	...(1)
Case II	$\lambda_1x^2+\lambda_2y^2+2rz=0,$	...(2)
Case III	$\lambda_1x^2+\lambda_2y^2+k=0,$	...(3)
Case IV	$\lambda_1x^2+2rz=0,$	...(4)
Case V	$\lambda_1x^2+k=0.$	...(5)

On comparison with the equations of the surfaces of revolution, we see that for the surface (1) to be that of revolution we must have two of $\lambda_1, \lambda_3, \lambda_3$ equal and for the surfaces (2) and (3) to be of revolution we must have $\lambda_1=\lambda_2$. The quadrics (4) and (5) cannot be surfaces of revolution.

Clearly the equation (1) will represent a sphere if the characteristic roots $\lambda_1, \lambda_2, \lambda_3$ are all equal.

Hence the result.

11·81. Conditions for the general equation of the second degree to represent a quadric of revolution. We have been in § 11·43 (B), p. 232 that if the discriminating cubic has two roots each equal to λ, then,

$$(b-\lambda)(c-\lambda)=f^2,\ (c-\lambda)(a-\lambda)=g^2,\ (a-\lambda)(b-\lambda)=h^2 \quad ...\text{I}$$
$$gh=(a-\lambda)f,\ hf=(b-\lambda)g,\ fg=(c-\lambda)\ h. \quad ...\text{II}$$

It can be shown that if these conditions are satisfied, then we can deduce the relations (4) and (14) of § 11·43 (*B*) p. 232 so that these conditions are sufficient also.

The required conditions will be obtained on eliminating λ.

11·82. *Firstly suppose that none of f, g, h is zero.* We can show that in this case the set of conditions I is deducible from II so that I is not an independent set of conditions and can, as such, be ignored. Let us assume the set II.

Now since

$$gh=(a-\lambda)f,$$

and

$$hf=(b-\lambda)g,$$

we get on multiplication

$$fgh^2=(a-\lambda)(b-\lambda)\,fg.$$

Dividing by $fg\neq 0$, we obtain

$$(a-\lambda)(b-\lambda)=h^2.$$

We may similarly deduce other conditions of the set I from II.

Now from II, we have

$$\lambda=a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h},$$

so that

$$a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h}$$

is the required set of conditions for the general equation of the second degree to represent a surface of revolution in case none of f, g, h is zero. These conditions can clearly be re-written as

$$F/f=G/g=H/h.$$

Cor. Assuming the conditions to be satisfied, we shall now obtain the equations of the axis of revolution.

Replacing

$$a, b, c$$

by

$$\lambda+\frac{gh}{f},\ \lambda+\frac{hf}{g},\ \lambda+\frac{fg}{y},$$

we get

$$F(x, y, z)\equiv\lambda(x^2+y^2+z^2)+fgh\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)^2+2ux+2vy+2wy+d$$

$$\equiv\lambda(x^2+y^2+z^2)+2ux+2vy+2wz+d+fgh\left(\frac{x}{f}+\frac{y}{g}+\frac{z}{h}\right)^2$$

This form of the equation shows that any plane parallel to the plane

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0 \qquad \ldots(1)$$

cuts the surface in a circle. The axis of revolution, being the locus of the centres of the circular sections, is the line through the centre of the sphere

$$\lambda(x^2+y^2+z^2)+2ux+2vy+2wz+d=0$$

perpendicular to the plane (1). Thus the axis of revolution is

$$\frac{x+\frac{u}{\lambda}}{1/f}=\frac{y+\frac{v}{\lambda}}{1/g}=\frac{z+\frac{w}{\lambda}}{1/h}.$$

11·83. We shall now consider the case when *some one of* f, g, h *is zero.*

Suppose that $f=0$, Then since

$$gh=(a-\lambda)f$$

we see that when $f=0$, we must have either $g=0$ or $h=0$.

Putting $f=0=g$ in I and II, we obtain

$$c=\lambda,\ (a-c)(b-c)=h^2$$

and taking $f=0=h$, we obtain

$$b=\lambda,\ (c-b)(a-b)=g^2.$$

Thus we have the alternative sets of conditions,

$$f=0=g,\ (a-c)(b-c)=h^2,\ \lambda=c \quad \ldots(1)$$

$$f=0=h,\ (c-b)(a-b)=g^2,\ \lambda=b \quad \ldots(2)$$

Starting with $g=0$, we shall obtain the alternative sets of conditions (1) and

$$g=0=h,\ (b-a)(c-a)=f^2,\ \lambda=a. \quad \ldots(3)$$

Thus if f, g, h are not all non-zero, we have three alternative sets of conditions (1), (2) and (3).

Axis of revolution. Suppose that the conditions (1) are satisfied. Since $(a-c)(b-c)=h^2$ we must have $a-c$, $b-c$ both of the same sign. Suppose that they are both positive.

We have

$$\begin{aligned} ax^2+by^2+cz^2+2fyz+2gzx+2hxy &= (a-c)x^2+(b-c)y^2+c(x^2+y^2+z^2)\pm 2\sqrt{[(a-c)(b-c)xy]} \\ &= [\sqrt{(a-c)}x\pm\sqrt{(b-c)}y]^2+c(x^2+y^2+z^2). \end{aligned}$$

Thus

$$\begin{aligned} F(x, y, z) = c(x^2+y^2+z^2)+2ux+2vy+2wz+d \\ +[\sqrt{(a-c)}x\pm\sqrt{(b-c)}y]^2 \end{aligned}$$

where the sign is + or − according as h is positive or negative.

Thus planes parallel to

$$\sqrt{(a-c)}x\pm\sqrt{(b-c)}y=0 \quad \ldots(4)$$

cut the surface in circular sections. Hence the axis of revolution is the line through the centre

$$(-u/c,\ -v/c,\ -w/c)$$

of the sphere

$$c(x^2+y^2+z^2)+2ux+2vy+2wz+d=0\ ;$$

perpendicular to (4). Thus the axis of revolution is

$$\frac{x+u/c}{\sqrt{(a-c)}}=\frac{y+v/c}{\pm\sqrt{(b-c)}},\ z+w/c=0.$$

The other alternatives may be similarly discussed.

11·9. Reduction of equations with numerical co-efficients. We shall now discuss the procedure to be followed when we have to reduce any given equation with numerical co-efficients to canonical form. It will be seen that when A, B, C, F, G, H are all zero, then we need not follow the method given in § 11·74 and instead obtain the required reduction by the method given below.

When A, B, C, F, G, H are all zero. We shall first show that *when A, B, C, F, G, H are all zero, then the second degree homogeneous part*

$$\Sigma(ax^2+2fyz)$$

must be a perfect square.

Since we have

$$bc=f^2,\ ca=g^2,\ ab=h^2,$$

we see that a, b, c must all be of the same sign. Without any loss of generality, we may suppose that a, b, c are all positive, for otherwise we could throughout multiply with -1 and have the same rendered positive.

Again, because

$$gh=af,\ hf=bg,\ fg=ch,$$

we see that f, g, h are either all positive or two negative, one positive.

Thus we have

$$\Sigma(ax^2+2fyz)=(\sqrt{a}x\pm\sqrt{b}y\pm\sqrt{c}z)^2,$$

so that the second degree terms form a perfect square.

Let the given equation be

$$(\sqrt{a}x+\sqrt{b}y+\sqrt{c}z)^2+2(ux+vy+wz)+d=0.$$

Case I. Suppose first that

$$\sqrt{a}:\sqrt{b}:\sqrt{c}=u:v:w$$

so that there exists a number k such that

$$\sqrt{a}=uk,\ \sqrt{b}=vk,\ \sqrt{c}=wk.$$

Then the given equation can be re-written as

$$k^2(ux+vy+wz)^2+2(ux+vy+wz)+d=0, \qquad \ldots(1)$$

so that the given equation represents a pair of parallel planes whose separate equations can be obtained on solving (1) as a quadric for $ux+vy+wz$.

Case II. Now suppose that the set of numbers $\sqrt{a}, \sqrt{b}, \sqrt{c}$ is not proportional to the set u, v, w so that

$$\sqrt{a}:\sqrt{b}\neq u:v \text{ or } \sqrt{b}:\sqrt{c}\neq v:w.$$

We re-write the given equation as

$$(\sqrt{a}x+\sqrt{b}y+\sqrt{c}z+\lambda)^2+2(u-\sqrt{a}\lambda)x+2(v-\sqrt{b}\lambda)y+2(w-\sqrt{c}\lambda)z+(d-\lambda^2)=0, \qquad \ldots(2)$$

and choose λ such that the two planes

$$\sqrt{a}x+\sqrt{b}y+\sqrt{c}z=0,$$

$$(u-\sqrt{a}\lambda)+x(v-\sqrt{b}\lambda)y+(w-\sqrt{c}\lambda)z=0$$

are perpendicular to each other. This requires

$$(u-\sqrt{a}\lambda)\sqrt{a}+(v-\sqrt{b}\lambda)\sqrt{b}+(w-\sqrt{c}\lambda)\sqrt{c}=0$$

i.e.,

$$u\sqrt{a}+v\sqrt{b}+w\sqrt{c}=\lambda(a+b+c)$$

or

$$\lambda=\frac{u\sqrt{a}+v\sqrt{b}+w\sqrt{c}}{a+b+c}, \text{ for } a+b+c\neq 0.$$

Having chosen λ, we re-write (2) as

$$\left(\frac{\sqrt{a}x+\sqrt{b}y+\sqrt{c}z+\lambda}{\sqrt{(a+b+c)}}\right)^2$$

$$=k\frac{2(\sqrt{a}\lambda-u)x+2(\sqrt{b}\lambda-v)y+2(\sqrt{c}\lambda-w)z+(\lambda^2-d)}{2\sqrt{[(\sqrt{a}\lambda-u)^2+(\sqrt{b}\lambda-v)^2+(\sqrt{c}\lambda-w)^2]}}$$

where

$$k=\frac{2[(\sqrt{a}\lambda-u)^2+(\sqrt{b}\lambda-v)^2+(\sqrt{c}\lambda-w)^2]}{a+b+c}.$$

Taking

$$\frac{\sqrt{a}x+\sqrt{b}y+\sqrt{c}z+\lambda}{\sqrt{(a+b+c)}}=Y$$

$$\frac{2(\sqrt{a}\lambda-u)x+2(\sqrt{b}\lambda-v)y+2(\sqrt{c}\lambda-w)x+\lambda^2-d}{2\sqrt{[(\sqrt{a}\lambda-u)^2+(\sqrt{b}\lambda-v)^2+(\sqrt{c}\lambda-w)^2]}}=X,$$

we see that the given equation takes the form

$$Y^2=kX,$$

so that the surface is a parabolic cylinder.

The following procedure is suggested for the reduction of numerical equations when the second degree terms do not form a perfect square.

1. Find the discriminating cubic and solve.

2. If no characteristic root is zero, then put down the centre giving equations and solve.

If (α, β, γ) is the centre and $\lambda_1, \lambda_2, \lambda_3$ are the characteristic roots, then the reduced equation is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+(u\alpha+v\beta+w\gamma+d)=0.$$

3. If one characteristic root is zero, find the principal direction l, m, n corresponding to the zero characteristic root by solving two of the three equations

$$al+hm+gn=0,$$
$$hl+bm+fn=0,$$
$$gl+fm+cn=0.$$

Then find $ul+vm+wn$. If this is not zero, the reduced equation is

$$\lambda_1x^2+\lambda_2y^2+2(ul+vm+wn)z=0;$$

λ_1, λ_2 being the non-zero characteristic roots.

4. If $ul+vm+wn=0$, find the centre giving equations. In this case we have a line of centres and only two of the three centre giving equations will be independent. Find any point (α, β, γ) satisfying two of the three equations. Then

$$\lambda_1x^2+\lambda_2y^2+(u\alpha+v\beta+w\gamma+d)=0$$

is the required reduced equation.

Note. If one characteristic root is zero and two non-zero, then the line of intersection of the two principal planes corresponding to the two non-zero roots is the axis, if the quadric is a paraboloid or an elliptic or hyperbolic cylinder and the line of intersection of the planes if the quadric is a pair of intersecting planes.

In the case of elliptic and hyperbolic cylinder, one pair of intersecting planes, the line of centres is also the axis.

Examples

1. *Reduce the equation*

$$2x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+2=0$$

to a canonical form.

The discriminating cubic is

$$\lambda^3+3\lambda^2-90\lambda+216=0.$$

This shows that $D=-216\neq 0$. The roots of the discriminating cubic are

$$3,\ 6,-12.$$

Again the centre giving equations are

$$2x-5y-4z+3=0,$$
$$5x+7y+5z-6=0,$$
$$4x+5y-2z+3=0.$$

Solving these we see that the centre is

$$(\tfrac{1}{3},\ -\tfrac{1}{3},\ \tfrac{4}{3}).$$

Denoting this by $(\alpha,\ \beta\ \gamma)$, we have

$$u\alpha+v\beta+w\gamma+d=-3.$$

Thus the canonical form of the equation is

$$3x^2+6y^2-12z^2-3=0$$

i.e.,

$$x^2+2y^2-4z^2=1, \qquad \ldots(1)$$

which shows that the given quadric is a hyperboloid of one sheet.

The equation (1) represents the given quadric when the origin of co-ordinates is its centre and the co-ordinate axes are parallel to the principal directions *i.e.*, (1) is an equation referred to principal axes as co-ordinate axes.

2. *Reduce to canonical form the equation of the quadric*

$$x^2-y^2+4yz+4xz-3=0.$$

The discriminating cubic is

$$\lambda^3-9\lambda=0$$

so that the characteristic roots are

$$0,\ 3,\ -3.$$

Thus $D=0$.

The direction cosines $l,\ m,\ n$ of the principal direction corresponding to $\lambda=0$ are given by

$$2l+4n=0,$$
$$-2m+4n=0,$$
$$4l+4m=0.$$

These give

$$l:m:n=2:-2:-1.$$

Thus in this case we have

$$ul+vm+wn=0,$$

so that we proceed to find the centre giving equations.

These are

$$2x+4z=0,$$
$$-2y+4z=0.$$
$$4y+4x=0.$$

These three planes meet in a line. Clearly (0, 0, 0) is a point on it. Denoting this by (α, β, λ), we have

$$u\alpha+v\beta+w\gamma+d=-3.$$

Thus the canonical form of the equation is

$$3x^2-3y^2=0,$$

i.e.,
$$x^2-y^2=0.$$

The given equation, therefore, represents a pair of intersecting planes.

Note. The fact that the given equation is free from first degree terms also shows that (0, 0, 0) is a centre of the given quadric.

3. *Show that*

$$2x^2+2y^2+z^2+2yz-2zx-4xy+x+y=0,$$

represents a paraboloid. Obtain its reduced equation. (*D.U. 1951*)

The discriminating cubic is

$$\lambda^3-5\lambda^2=2\lambda=0$$

Its roots are

$$0, \frac{5+\sqrt{21}}{2}, \frac{5-\sqrt{21}}{2}.$$

This shows that $D=0$. The direction cosines l, m, n of the principal direction corresponding to $\lambda=0$ are given by

$$4l-4m-2n=0, \quad \ldots(1)$$
$$-4l+4m+2n=0, \quad \ldots(2)$$
$$-2l+2m+2n=0. \quad \ldots(3)$$

Clearly (1) and (2) are the same. Solving (2) and (3), we obtain

$$l=\frac{1}{\sqrt{2}}, m=\frac{1}{\sqrt{2}}, n=0.$$

$\therefore$
$$ul+vm+wn=\frac{1}{\sqrt{2}}\neq 0.$$

Thus the reduced equation is

$$\frac{5+\sqrt{21}}{2}x^2+\frac{5-\sqrt{21}}{2}y^2+\sqrt{2}z=0.$$

4. *Discuss the nature of the surface whose equation is*

$$4x^2-y^2-z^2+2yz-8x-4y+8z-2=0$$

and find the co-ordinates of its vertex and equations to its axis.
(*Lucknow, 1949*)

It may be shown that the roots of the discriminating cubic are

$$0, -2, 4.$$

The direction cosines l, m, n of the principal direction corresponding to the root, 0, are given by

$$8l=0,$$
$$-2m+2n=0,$$
$$-2m+2n=0.$$

These give $l=0,\ m=1/\sqrt{2},\ n=1/\sqrt{2}.$

Then $$ul+vm+wn=\frac{2}{\sqrt{2}}\neq 0.$$

Thus the quadric is a paraboloid.

We now proceed to find the axis and the vertex.

The direction cosines l, m, n, of the principal direction corresponding to $\lambda=-2$ are given by

$$6l+0m+0n=0,$$
$$0l+m+n=0,$$
$$0l+m+n=0.$$

These give $$l=0,\ m=\frac{1}{\sqrt{2}},\ n=-\frac{1}{\sqrt{2}},$$

so that the corresponding principal plane is

$$-2(y-z)+(-2-4)=0,$$

i.e., $$y-z+3=0. \qquad \ldots(1)$$

Again the direction cosines of the principal direction corresponding to $\lambda=4$ are given by

$$0l+0m+0n=0,$$
$$0l-5m+n=0,$$
$$0l+m-5n=0.$$

These give

$$l:m:n=1:0:0$$

so that the corresponding principal plane is

$$4x-4=0$$

i.e.,

$$x=1. \qquad \ldots(2)$$

Thus

$$y-z+3=0,\ x=1,$$

is the required axis of the paraboloid.

The vertex is the point where the axis meets the paraboloid. Re-writing the equations of the axis in the form

$$\frac{x-1}{0}=\frac{y+3}{1}=\frac{z}{1},$$

we see that any point

$$(1,\ r-3,\ r),$$

on the axis will lie on the surface for

$$r=\tfrac{3}{4},$$

so that *the vertex is the point*

$$(1,\ -\tfrac{9}{4},\ \tfrac{3}{4}).$$

5. *Prove that*

$$5x^2+5y^2+8z^2+8yz+8zx-2xy+12x-12y+6=0$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$.

Find also the equations of the axis of the cylinder. (*Calcutta, 1953*)

The discriminating cubic is

$$\lambda^3-18\lambda^2+72\lambda=0$$

so that the values of λ are

$$0,\ 6,\ 12.$$

The direction cosine l, m, n of the principal direction corresponding to $\lambda=0$ are given by

$$l-5m-4n=0$$
$$5l-m+4n=0$$

so that

$$l=\frac{1}{\sqrt{3}},\ m=\frac{1}{\sqrt{3}},\ n=-\frac{1}{\sqrt{3}}$$

Thus

$$ul+vm+wn=\frac{6}{\sqrt{3}}-\frac{6}{\sqrt{3}}-\frac{0}{\sqrt{3}}=0.$$

We have, therefore, to proceed to put down centre giving equations. These are

$$10x-2y+8z+12=0, \quad \ldots(1)$$
$$-2x+10y+8z-12=0, \quad \ldots(2)$$
$$8x+8y+16z=0. \quad \ldots(3)$$

Clearly (3) can be obtained on adding (1) and (2) so that as expected, these three equations are equivalent to only two. Putting $z=0$ in (1) and (2), we obtain

$$x=-1,\ y=1,\ z=0,$$

so that $(-1, 1, 0)$ is a centre. Thus

$$u\alpha+v\beta+w\gamma+d=-6-6+6=-6.$$

Hence the reduced equation is

$$12x^2+6y^2-6=0,$$

i.e.,

$$2x^2+y^2=1.$$

The cross-section is

$$2x^2+y^2=1,\ z=0.$$

Its eccentricity is now easily seen to be $1/\sqrt{2}$.

The line of centres is the axis of the cylinder so that the equations of the axis are

$$5x-y+4z+6=0,\ x+y+2z=0.$$

6. *Show that the equation*

$$x^2+2yz=1$$

represents a quadric of revolution and find the axis of revolution.

The discriminating cubic is

$$(1-\lambda)(\lambda^2-1)=0,$$

i.e.,
$$(\lambda+1)(\lambda-1)^2=0,$$

so that the characteristic roots are

$$-1,\ 1,\ 1.$$

Two of the characteristic roots being equal, we see that the given equation represents a quadric of revolution.

Further re-writing the equation as

$$(x^2+y^2+z^2)-(y-z)^2=1.$$

i.e.,
$$(x^2+y^2+z^2-1)-(y-z)^2=0,$$

we see that the planes parallel to

$$y-z=0 \qquad \ldots(1)$$

cut the quadric in circles. Thus the axis of revolution which is the line through the centre of the sphere

$$x^2+y^2+z^2=1$$

perpendicular to the line (1) is

$$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{1},$$

i.e.,
$$x=0,\ y=z.$$

7. *Prove that*

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$$

represents a paraboloid of revolution and find the co-ordinates of its focus. (*D.U. 1954*)

The discriminating cubic is

$$-4\lambda^3+12\lambda^2-9\lambda=0$$

so that the characteristic roots are

$$0,\ \tfrac{3}{2},\ \tfrac{3}{2}.$$

Two values of λ being equal, the given quadric is a surface of revolution.

The direction cosines l, m, n of the principal direction corresponding to $\lambda=0$ are given by any two of the three equations

$$l-\tfrac{1}{2}m-\tfrac{1}{2}n=0,$$
$$-\tfrac{1}{2}l+m-\tfrac{1}{2}n=0,$$
$$-\tfrac{1}{2}l-\tfrac{1}{2}m+n=0.$$

These give

$$l:m:n=1:1:1.$$

$\therefore$
$$l=1/\sqrt{3},\ m=1/\sqrt{3},\ n=1/\sqrt{3},$$

Now we have

$$ul+vm+wn=-\frac{3}{2}\cdot\frac{1}{\sqrt{3}}-3\cdot\frac{1}{\sqrt{3}}-\frac{9}{2}\cdot\frac{1}{\sqrt{3}}$$

$$=-\frac{9}{\sqrt{3}}\neq 0,$$

Thus the quadric is a paraboloid of revolution and the reduced equation is

$$\frac{3}{2}x^2+\frac{3}{2}y^2-2.\frac{9}{\sqrt{3}}z=0,$$

i.e.,

$$x^2+y^2=4\sqrt{3}z.$$

This form of the equation shows that the latus rectum of the generating parabola is $4\sqrt{3}$.

With respect to the given system of co-ordinate axes, the direction ratios of the axis of the paraboloid which is also the axis of revolution are

$$1, 1, 1.$$

We re-write the given equations in the form

$$x^2+y^2+z^2-\tfrac{1}{2}[(x+y+z)^2-(x^2+y^2+z^2)]-3x-6y-9z+21=0$$

i.e.,

$$\tfrac{3}{2}(x^2+y^2+z^2)-3x-6y-9z+21-\tfrac{1}{2}(x+y+z)^2=0$$

or

$$x^2+y^2+z^2-2x-4y-6z+14-\tfrac{1}{3}(x+y+z)^2=0.$$

Thus the axis of revolution, being the line through the centre of the sphere

$$x^2+y^2+z^2-2x-4y-6z+14=0,$$

and perpendicular to the plane

$$x+y+z=0,$$

is

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}, \qquad \ldots(1)$$

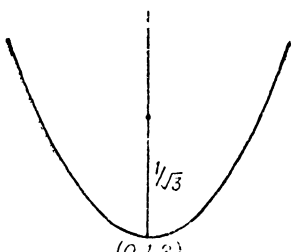

Fig. 32

which is the axis of the paraboloid.

The vertex is the point where this axis meets the paraboloid. It can be shown that any point

$$(r+1, r+2, r+3)$$

on the axis will be on the paraboloid if

$$r=-1.$$

Thus (0, 1, 2) is the vertex of the paraboloid.

The required focus is the point on the axis (1) at a distance $\sqrt{3}$ from (0, 1, 2). Re-writing the equations of the axis in the form

$$\frac{x-0}{1/\sqrt{3}}=\frac{y-1}{1/\sqrt{3}}=\frac{z-2}{1/\sqrt{3}},$$

we see that the point on the axis at a distance $\sqrt{3}$ from (0, 1, 2) is

$$(1, 2, 3).$$

Thus (1, 2, 3) is the required focus.

8. *If*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

represents a pair of planes, prove that the planes bisecting the angles between them are

$$\begin{vmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \\ x & y & z \\ F^{-1} & G^{-1} & H^{-1} \end{vmatrix}=0.$$

As the given equation represents a pair of planes, we must have $D=0$.

The line of intersection of the two planes is parallel to the principal direction corresponding to the characteristic root zero so that if l, m, n be the direction cosines of this line we have

$$al+hm+gn=0,$$
$$hl+bm+fn=0.$$

These give

$$\frac{l}{G}=\frac{m}{F}=\frac{n}{C}.$$

As $FG=CH$ we see on replacing C by FG/H, that l, m, n are proportional to F^{-1}, G^{-1}, H^{-1}.

This result can also be obtained if we regard the line of intersection as the line of centres.

Now we know that the two bisecting planes are the principal planes corresponding to the two non-zero characteristic roots.

Suppose that (x, y, z) is any point on either bisecting plane. Let this bisecting plane, as a principal plane, bisect chords with direction cosines l_1, m_1, n_1 and perpendicular to the plane. The equation of the plane being

$$l_1(ax+hy+gz)+m_1(hx+by+fz)+n_1(gx+fy+cz)=0, \quad \ldots(1)$$

we see that any point (x, y, z) on the bisecting plane satisfies this equation.

Further the plane being normal to the line with direction cosines l_1, m_1, n_1, its equation is also

$$l_1x+m_1y+n_1z=0, \quad \ldots(2)$$

so that (x, y, z) satisfies (2) also.

Finally, the principal direction l_1, m_1, n_1 corresponding to a non-zero characteristic root being perpendicular to that corresponding to the zero characteristic root, we have

$$l_1F^{-1}+m_1G^{-1}+n_1H^{-1}=0. \qquad \ldots(3)$$

From (1), (2) and (3), we have

$$\begin{vmatrix} ax+hy+gz & hx+by+fz & gx+fy+cz \\ x & y & z \\ F^{-1} & G^{-1} & H^{-1} \end{vmatrix}=0.$$

Hence the result.

9. *Prove that if*

$$a^3+b^3+c^3=3abc \text{ and } u+v+w\neq 0,$$

$$ax^2+by^2+cz^2+2ayz+2bzx+2cxy+2ux+2vy+2wz+d=0$$

represents either a parabolic cylinder or a hyperbolic paraboloid.

(*D. U. 1952*)

The discriminating cubic of the given quadric is

$$\lambda^3-\lambda^2(a+b+c)+\lambda(ab+bc+ca-a^2-b^2-c^2)-(3abc-a^3-b^3-c^3)=0,$$

so that under one of the given conditions, one root is zero.

We have

$$0=a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-cb-ca)$$

so that either

$$a+b+c=0, \qquad \ldots(1)$$

or

$$a^2+b^2+c^2-ab-bc-ca=0. \qquad \ldots(2)$$

The condition (2) is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2=0$$

i.e.,

$$a=b=c. \qquad \ldots(3)$$

Assuming (2) to be satisfied, we see that the given equation takes the form

$$a(x+y+z)^2+2(ux+vy+wz)+d=0$$

which is a parabolic cylinder, if

$$u\neq v \quad \text{or} \quad v\neq w.$$

Suppose now that the condition (1) is satisfied so that one root only of the discriminating cubic is zero.

The direction cosines l, m, n of the principal direction corresponding to the zero root are given by

$$al+cm+bn=0,$$
$$cl+bm+an=0,$$

so that

$$\frac{l}{ac-b^2}=\frac{m}{bc-a^2}=\frac{n}{ab-c^2}.$$

As

$a+b+c=0$, we may see that

$$ac-b^2=bc-a^2=ab-c^2.$$

Thus the principal direction corresponding to the zero root is given by

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Also

$$ul+vm+wn=\frac{1}{\sqrt{3}}(u+v+w)\neq 0.$$

Thus in this case the quadric is a paraboloid. This paraboloid is hyperbolic for the two non-zero characteristic roots given by

$$\lambda^2+(ab+bc+ca-a^2-b^2-c^2)=0$$

are of opposite signs.

Exercises

1. Show that

$$4x^2-y^2-z^2+2yz-8x-4y+8z-2=0$$

represents a paraboloid. Find the reduced equation and the co-ordinates of the vertex. (*Lucknow 1952*)

2. Reduce to its principal axes :

$$2y^2-2yz+2zx-2xy-x-2y+3z-2=0$$

and state the nature of the surface represented by the equation.

(*Lucknow, Hons. 1952*)

3. Find the nature of the surface represented by the equation

$$x^2+2y^2-3z^2-4yz+8zx-12xy+1=0.$$

(*P.U. 1949*)

4. Find the reduced equation of

(*i*) $x^2+2yz-4x+6y+2z=0.$

(*ii*) $x^2-y^2+2yz-2xz-x-y+z=0.$

(*iii*) $yz+zx+xy-7x-6y-5z-25=0.$

(*iv*) $4y^2-4yz+4zx-4xy-2x+2y-1=0.$

(*v*) $2x^2+2y^2+z^2+2yz-2zx-4xy+x+y=0.$ (*Lucknow 1947*)

(*vi*) $(x\cos\alpha-y\sin\alpha)^2+(y\cos\alpha+z\sin\alpha)^2+2y=1.$

(*vii*) $3x^2+6yz-y^2-z^2-6x+6y-2z-2=0,$ (*M.U. 1947*)

(*viii*) $4x^2+y^2+z^2-4xy-2yz+4zx-12x+6y-6z+8=0.$

(*ix*) $x^2+y^2+z^2-2xy-2yz+2zx+x-4y+z+1=0.$

5. Show that the equation

$$a(z-x)(x-y)+b(x-y)(y-z)+c(y-z)(z-x)=0,$$

represents two planes whose line of intersection is equally inclined to the three co-ordinate axes.

6. Show that the equation

$$2yz+2zx+2xy=1$$

represents a hyperboloid of revolution. Is this an hyperboloid of one or two sheets ?

7. Show that the quadric

$$2y^2+4zx-6x-8y+2z+5=0.$$

is a cone and obtain its reduced equation. Show further that this is a right circular cone with its axis of revolution parallel to the line

$$x+z=0=y.$$

8. Show that the quadric with generators

$$y=1,\ z=-1\ ;\ z=1,\ x=-1\ ;\ x=1,\ y=-1$$

is a hyperboloid of revolution.

9. Find the reduced equation of the quadric with generators

$$x-1=0=y-1,$$
$$x=0=y-z,$$
$$x-2=0=z.$$

10. Prove that every quadric of the linear system determined by the two equations

$$y^2-zx+x=0,\ x^2+y^2+2xz=0$$

is a cone.

11. Discuss the nature of the quadrics represented by the equation

$$2x^2+(m^2+2)(y^2+z^2)-4(yz+zx+xy)=m^2-2m+2$$

as m varies from $-\infty$ to $+\infty$.

Obtain the reduced equation of the quadric corresponding to $m=1$,

12. Show that there is only one paraboloid in the system of quadrics

$$\Sigma(ax^2-2fyz)+2\Sigma ux+d+\lambda[lx+my+nz+p]^2=0.$$

In particular, show that if f, g, h, u, v, w are all zero, the equation of this paraboloid is

$$ax^2+by^2+cz^2+d\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)-(lx+my+nz+p)^2=0.$$

Further prove that its axis is parallel to the line

$$\frac{ax}{l}=\frac{by}{m}=\frac{cz}{n}.$$

13. If the general equation

$$\Sigma(ax^2+2fyz)+2\Sigma ux+d=0,$$

represents a right circular cylinder, prove that

$$\frac{a}{f}+\frac{h}{g}+\frac{g}{h}=0\ ;\ \frac{h}{f}+\frac{b}{g}+\frac{f}{h}=0\ ;\ \frac{g}{f}+\frac{f}{g}+\frac{c}{h}=0\ ;$$

$$\frac{u}{f}+\frac{v}{g}+\frac{w}{h}=0. \qquad (M.U.\ 1953)$$

14. Show that the condition for the quadric

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$$

to be a cone is

$$\frac{\alpha^2}{a^2+\lambda}+\frac{\beta^2}{b^2+\lambda}+\frac{\gamma^2}{c^2+\lambda}=1.$$

15. Prove that the principal axes of the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

are given by the equations

$$x(f\lambda_r+F)=y(g\lambda_r+G)=z(h\lambda_r+H),\ (r=1, 2, 3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0$$

and $F=gh-af,\ G=hf-bg,\ H=fg-ch.$

Also show that the cone which touches the co-ordinate planes and the principal planes of the above conicoid is

$$\sqrt{[(gH-hG)x]}+\sqrt{[(hF-fH)y]}+\sqrt{(fG-gF)z}]=0. \qquad (B.U.\ 1953)$$

16. If the feet of the six normals from P to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lie upon a concentric conicoid of revolution, prove that the locus of P is the cone

$$\frac{y^2z^2}{a^2(b^2-c^2)}+\frac{z^2x^2}{b^2(c^2-a^2)}+\frac{x^2y^2}{c^2(a^2-b^2)}=0$$

and that the axes of symmetry of the conicoids lie on the cone

$$a^2(b^2-c^2)x^2+b^2(c^2-a^2)y^2+c^2(a^2-b^2)z^2=0 \qquad (B.U.\ 1953)$$

17. Prove that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

will represent a right circular cone whose verticle angle is θ, provided that

$$\frac{af-gh}{f}=\frac{bg-hf}{g}=\frac{ch-fg}{h}=\frac{(a+b+c)(1+\cos\theta)}{(1+3\cos\theta)}$$

18. Given the ellipsoid of revolution

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1,$$

$(a^2>b^2)$, show that the cone whose vertex is one of the foci of the ellipse $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and whose base is any plane section of the ellipsoid is a surface of revolution. $(D.U.\ 1948)$

19. Prove that if

$$F(x, y, z)\equiv\Sigma(ax^2+2fyz)+2\Sigma ux+d=0,$$

represents a paraboloid of revolution, we have

$$agh+f(g^2+h^2)=bhf+g(h^2+f^2)=cfg+h(f^2+g^2)=0$$

and that if it represents a right circular cylinder, we have also

$$\frac{u}{f}+\frac{v}{g}+\frac{w}{h}=0. \qquad (D.U.\ 1950, 53)$$

Answers

1. $2x^2-y^2+\sqrt{2}z=0$. $\left(1, -\frac{9}{4}, \frac{3}{4}\right)$.

2. $3x^2-y^2=\frac{1}{2}$. Hyperbolic cylinder.

3. $3x^2+6y^2-9z^2+1=0$. Hyperboloid of the two sheets.

4. (*i*) $x^2+y^2-z^2=10$.

(*ii*) $3x^2-3y^2=z$.

(*iii*) $2x^2-y^2-z^2=102$.

(*iv*) $6x^2-2y^2=1$.

(*v*) $\frac{5+\sqrt{17}}{2}x^2+\frac{5-\sqrt{17}}{2}y^2+\sqrt{2}z=0$.

(*vi*) $(1+\sin\alpha\cos\alpha)x^2+(1-\sin\alpha\cos\alpha)y^2+z\sin 2\alpha/\sqrt{(1-\sin^2\alpha\cos^2\alpha)}=0$. if $\sin\alpha\neq 0$, $\cos\alpha\neq 0$.

$x^2+y^2=2$ if $\sin\alpha=0$ and $y^2+z^2=2$ if $\cos\alpha=0$.

(*vii*) $2x^2+3y^2-4z^2=4$.

(*viii*) $3x^2-3\sqrt{6}x+4=0$.

(*ix*) $3y^2=\sqrt{6}x$.

6. Hyperboloid of two sheets.

7. $x^2=y^2+z^2$.

9. $\frac{\sqrt{6}+1}{2}x^2-\frac{\sqrt{6}-1}{2}y^2+\frac{2}{\sqrt{5}}z=0$.

11. For $m>2$, ellipsoid
For $m<-2$, ellipsoid
For $m=2$, pair of imaginary planes
For $m=-2$, elliptic cylinder
For $1<m<2$, hyperboloid of two sheets
For $-2<m<1$, hyperboloid of one sheet
For $m=1$, cone
The reduced equation for the last case is

$$10x^2+(3+\sqrt{33})y^2=(\sqrt{33}-3)z^2.$$

APPENDIX

Spherical Polar and Cylindrical Coordinates.

Various systems of coordinates have been devised to meet different types of problems which arise in Geometry and in various applications of the same. Cartesian system which is one of these has already been introduced and this is the one system with which we have been concerned all along. It is now proposed to introduce two more systems, *viz.* :

1. *Cylindrical Polar,*
2. *Spherical Polar*

which are often found useful in various applications.

Cylindrical Polar Coordinates.

Let P be any given point.

Draw PN perpendicular to the XY-plane, N being the foot of the perpendicular.

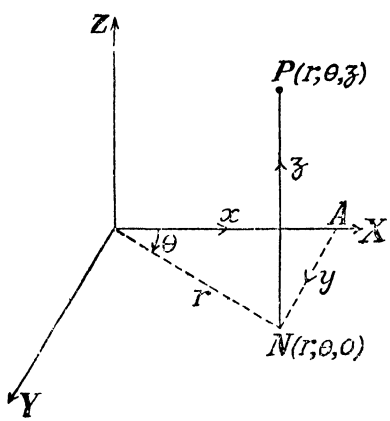

Fig. 33

We write

$ON=r$, $\angle XON=\theta$, $NP=z$.

Then r, θ, z are called the cylindrical polar coordinates of the point P.

It will be seen that r, θ are the usual polar coordinates of the projection N in the XY-plane of the point P referred to O as the pole and OX as the initial line.

If x, y, z be the cartesian coordinates of P referred to OX, OY, OZ as the three axes, we may easily obtain the following formulae giving relations between x, y, z and r, θ, z.

$$x=r\cos\theta,\ y=r\sin\theta,\ z=z.$$

Ex. What are the surfaces represented by

(*i*) $r=$constant ;

(*ii*) $\theta=$constant ;

(*iii*) $z=$constant.

Spherical Polar Coordinates.

Let N be the foot of the perpendicular from P on the XY-plane. We write

$$OP=r,\ \angle POZ=\theta,\ \angle XON=\phi$$

It may be easily seen that ϕ can also be described as the angle between the planes

$$POZ \text{ and } XOZ.$$

Then r, θ, ϕ are known as the spherical polar coordinates of P.

We now obtain the formulae of transformation between x, y, z, and r, θ, ϕ.

Draw $NA \perp OX$

We have $\angle OPN = \theta$.

From the right-angled triangle OPN, we have

$$z = NP = OP \cos\theta$$
$$= r\cos\theta.$$
$$ON = OP \sin \angle OPN$$
$$= r \sin\theta$$

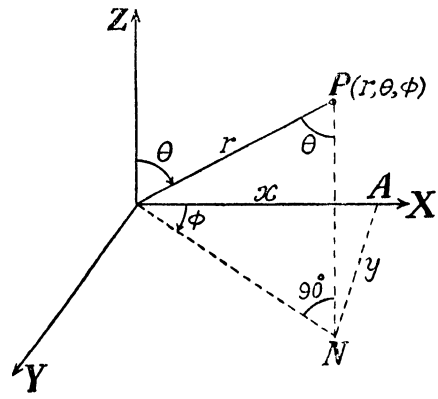

Fig. 34

Again, from the right angled triangle OAN, we have

$$x = OA = ON\cos\phi = r\sin\theta\cos\phi$$
$$y = NA = ON\sin\phi = r\sin\theta\sin\phi$$

Thus we have the following formulae of transformation.

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

Surfaces represented by

(*i*) r = constant ;

(*ii*) θ = constant ;

(*iii*) ϕ = constant.

The reader may easily verify that

(*i*) r = constant represents a sphere with its centre at the origin,

(*ii*) θ = constant represents a right circular cone with its vertex at the origin and OZ as its axis,

(*iii*) ϕ = constant represents a semi-plane through OZ.

It may be easily verified that if a point r, θ, ϕ varies in the interior of a sphere whose centre is at the origin and the radius is a : then r varies from 0 to a ; ϕ varies from 0 to 2π ; θ varies from 0 to π.

INDEX

(Numbers refer to pages)